# Second-Order Nédélec Curl-Conforming Prism for Finite Element Computations

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# Motivation: Why another triangular prism?

- Previous approaches:
  - Volakis, 1997.
  - Graglia, 1998.
  - Tsiboukis, 2008.
  - Jiao, 2009.
  - Tobon, 2014.
- Systematic approach:

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  - *A priori* known space of functions.
  - Definition of degrees of freedom as functionals.
  - Basis functions as dual basis with respect to those degrees of freedom.
- Compatibility with tetrahedra previously implemented.

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# Outline

- Definition.
- Verification.
- Comparison with other authors: condition number.
- Integration in HOFEM (*Higher Order Finite Element Method*).

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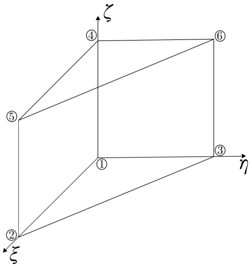
- Domain.



# Space construction

- Tensor product between triangle and segment.

$$\boldsymbol{\mathcal{P}}_{\boldsymbol{k}}^{\text{prism}} = (\mathcal{R}^k(\widehat{T}) \otimes \mathcal{P}_k(\widehat{I})) \times (\mathcal{P}_k(\widehat{T}) \otimes \mathcal{P}_{k-1}(\widehat{I}))$$

- Space dimension.

| Dimensions | $\mathcal{R}^k(\widehat{T})$ | $\mathcal{P}_k(\widehat{I})$ | $\mathcal{P}_k(\widehat{T})$ | $\mathcal{P}_{k-1}(\widehat{I})$ | Total |
|---|---|---|---|---|---|
| $k$ | $(k+2)k$ | $k+1$ | $\frac{(k+1)(k+2)}{2}$ | $k$ | — |
| $k=1$ | 3 | 2 | 3 | 1 | 9 |
| $k=2$ | 8 | 3 | 6 | 2 | 36 |
| $k=3$ | 15 | 4 | 10 | 3 | 90 |
| $k=4$ | 24 | 5 | 15 | 4 | 180 |

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# Space definition

- First order.

$$\mathcal{P}_1^{\text{prism}} \equiv \mathbf{N}_i(i = 1, ..., 9) = \left\{ \begin{array}{c} a_1^{(i)} + a_2^{(i)} z + C^{(i)} y + D^{(i)} yz \\ b_1^{(i)} + b_2^{(i)} z - C^{(i)} x - D^{(i)} xz \\ c_1^{(i)} + c_2^{(i)} x + c_3^{(i)} y \end{array} \right\}$$

- Second order.

$$\mathcal{P}_2^{\text{prism}} \equiv \mathbf{N}_i(i = 1, ..., 36) = \left\{ \begin{array}{c} a_1^{(i)} + a_2^{(i)} x + a_3^{(i)} y + a_4^{(i)} z + a_5^{(i)} xz + a_6^{(i)} yz + a_7^{(i)} z^2 + a_8^{(i)} xz^2 + ... \\ ... + a_9^{(i)} yz^2 + C^{(i)} y^2 + D^{(i)} xy + E^{(i)} y^2 z + F^{(i)} xyz + G^{(i)} y^2 z^2 + H^{(i)} xyz^2 \\ b_1^{(i)} + b_2^{(i)} x + b_3^{(i)} y + b_4^{(i)} z + b_5^{(i)} xz + b_6^{(i)} yz + b_7^{(i)} z^2 + b_8^{(i)} xz^2 + ... \\ ... + b_9^{(i)} yz^2 - C^{(i)} xy - D^{(i)} x^2 - E^{(i)} xyz - F^{(i)} x^2 z - G^{(i)} xyz^2 - H^{(i)} x^2 z^2 \\ c_1^{(i)} + c_2^{(i)} x + c_3^{(i)} y + c_4^{(i)} x^2 + c_5^{(i)} y^2 + c_6^{(i)} xy + c_7^{(i)} z + c_8^{(i)} xz + ... \\ ... + c_9^{(i)} yz + c_{10}^{(i)} x^2 z + c_{11}^{(i)} y^2 z + c_{12}^{(i)} xyz \end{array} \right\}$$

# Definition of the degrees of freedom

- Edges.

$$g(\mathbf{u}) = \int_e (\mathbf{u} \cdot \hat{\boldsymbol{\tau}}) q \, dl, \forall q \in P_1(e)$$

- Triangular faces.

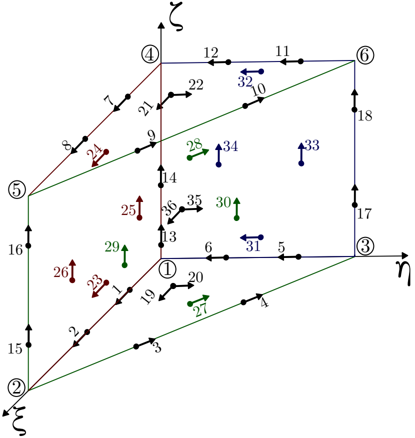
$$g(\mathbf{u}) = \int_{f_t} (\mathbf{u} \times \hat{\mathbf{n}}) \cdot \mathbf{q} \, ds, \forall \mathbf{q} \in \mathbf{P}_0(f_t)$$

- Quadrilateral faces.

$$g(\mathbf{u}) = \int_{f_q} (\hat{\mathbf{n}} \times \mathbf{u}) \cdot \mathbf{q} \, ds, \forall \mathbf{q} = (q_1, q_2); q_1 \in \mathcal{Q}_{0,1}; q_2 \in \mathcal{Q}_{1,0}$$

- Volume.

$$g(\mathbf{u}) = \int_V \mathbf{u} \cdot \mathbf{q} \, dV, \forall \mathbf{q} \in \mathbf{P}_0(f_t)$$


ζ
η
ξ
① ② ③ ④ ⑤ ⑥
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36

# Other considerations (i)

- Discretization: choice of $\mathbf{q}$.

**Dual basis**

$$g_i(\mathbf{N_j}) = \delta_{ij}$$

- $a_1, a_2, \ldots$ as unknowns.

| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $\ldots$ |
|---|---|---|---|---|---|---|---|---|---|
| 4 | −6 | −12 | −16 | 24 | 48 | 12 | −18 | −36 | $\ldots$ |
| −2 | 6 | 2 | 8 | −24 | −8 | −6 | 18 | 6 | $\ldots$ |
| 0 | 0 | 2 | 0 | 0 | −8 | 0 | 0 | 6 | $\ldots$ |
| 0 | 0 | 4 | 0 | 0 | −16 | 0 | 0 | 12 | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

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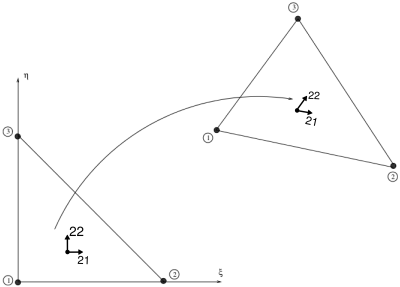
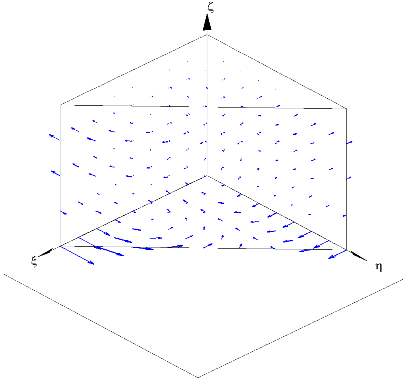
# Other considerations (& ii)

- Local definition of $\hat{\boldsymbol{\tau}}$, $\hat{\mathbf{n}}$, $\mathbf{q}$.
- Use of the master element:

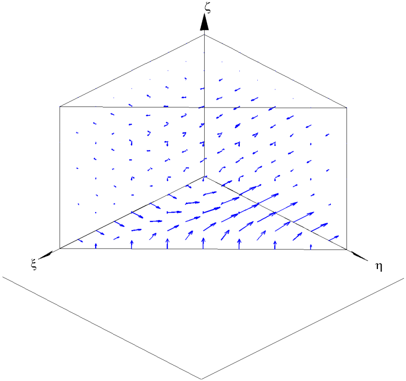
$$\mathbf{u} = [J]^{-1}\widehat{\mathbf{u}}$$



# Basis functions on triangular faces

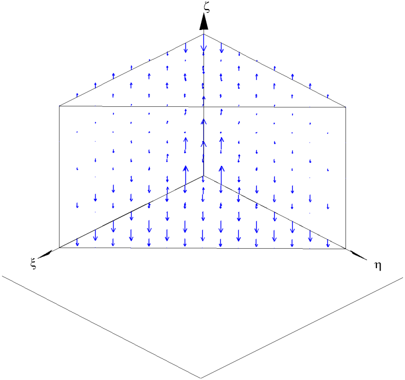



$\mathbf{N}_3$

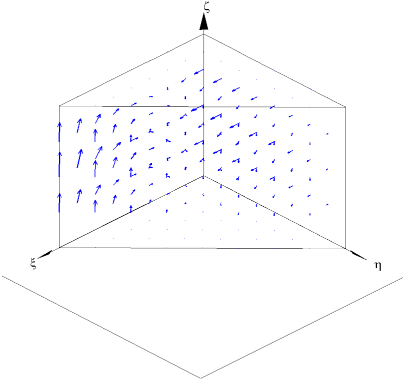



$\mathbf{N}_{19}$

# Basis functions on quadrilateral faces




$\mathbf{N}_{13}$




$\mathbf{N}_{24}$

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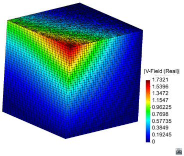
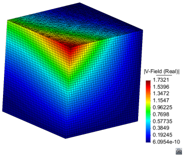
# Verification: MMS

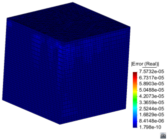
- $\nabla \times \frac{1}{\mathbf{f_r}} \nabla \times \mathbf{u} - k_0^2 \boldsymbol{g_r} \mathbf{u} = \boldsymbol{\Psi}.$
- HOFEM: Monomials $(xyz^2, -xz^2, xyz)$.




MMS solution




Code solution

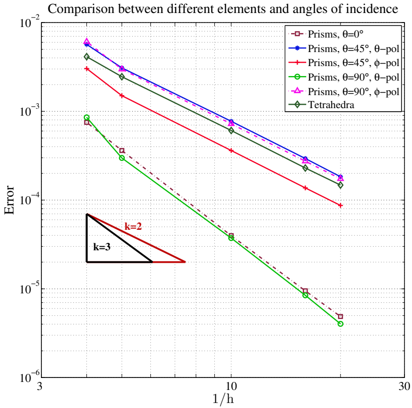



Error

- HOFEM: Planewave.

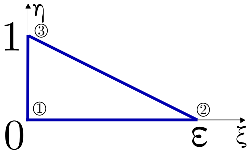
# Verification: MMS

- $\nabla \times \frac{1}{\mathbf{f_r}} \nabla \times \mathbf{u} - k_0^2 \boldsymbol{g_r} \mathbf{u} = \boldsymbol{\Psi}$.
- HOFEM: Monomials $(xyz^2, -xz^2, xyz)$.
- HOFEM: Planewave.



# Triangle deformation

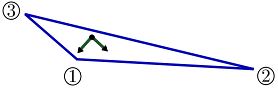
$$[M^p] = [D]^{-1}[M][D]^{-1}$$
$$[K^p] = [D]^{-1}[K][D]^{-1}$$
$$D_{ii} = \sqrt{M_{ii}}$$



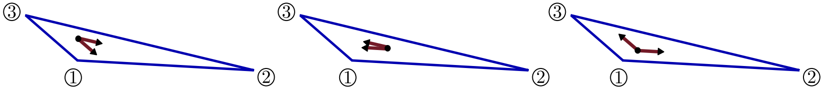

| | Reference prism | | Triangle deformation | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $\varepsilon = 4$ | | $\varepsilon = 8$ | | $\varepsilon = 16$ | |
| Version | $[M^p]$ | $[K^p]$ | $[M^p]$ | $[K^p]$ | $[M^p]$ | $[K^p]$ | $[M^p]$ | $[K^p]$ |
| *vc*,(1-2) | 81 | 37 | 1587 | 210 | 18826 | 791 | 276385 | 3096 |
| *vc*,(2-3) | 81 | 37 | 217 | 199 | 738 | 733 | 2827 | 2856 |
| *vc*,(3-1) | 71 | 38 | 215 | 197 | 737 | 732 | 2825 | 2854 |
| *vq* | 72 | 37 | 215 | 197 | 737 | 732 | 2826 | 2854 |
| Graglia | 37 | 19 | 174 | 104 | 639 | 394 | 2498 | 1551 |
| Tobon | 171 | 20 | 842 | 101 | 3468 | 398 | 14046 | 1588 |

# Different options in q

vq strategy




vc strategy

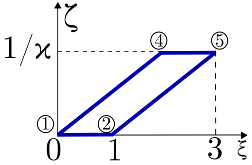


# Rectangle deformation

$$[M^p] = [D]^{-1}[M][D]^{-1}$$

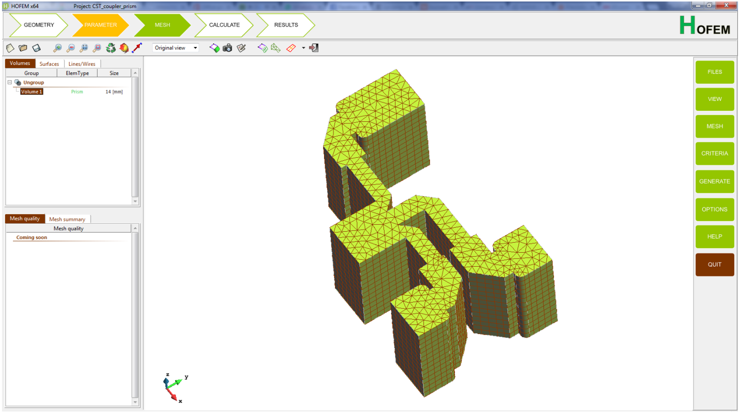
$$[K^p] = [D]^{-1}[K][D]^{-1}$$

$$D_{ii} = \sqrt{M_{ii}}$$




| | Reference prism | | Rectangle deformation | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $\kappa = 2$ | | $\kappa = 4$ | | $\kappa = 8$ | |
| Version | $[M^p]$ | $[K^p]$ | $[M^p]$ | $[K^p]$ | $[M^p]$ | $[K^p]$ | $[M^p]$ | $[K^p]$ |
| *vc* | 72 | 37 | 3107 | 2566 | 12270 | 10205 | 48926 | 40765 |
| *vq* | 72 | 37 | 2187 | 2066 | 8435 | 8171 | 33432 | 32599 |
| Graglia | 37 | 19 | 1484 | 1067 | 5889 | 4279 | 23509 | 17131 |
| Tobon | 171 | 20 | 5967 | 1209 | 23559 | 4226 | 93928 | 16923 |

# Comparison (i)

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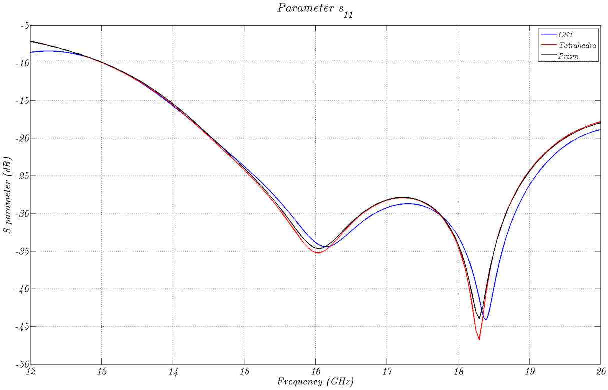
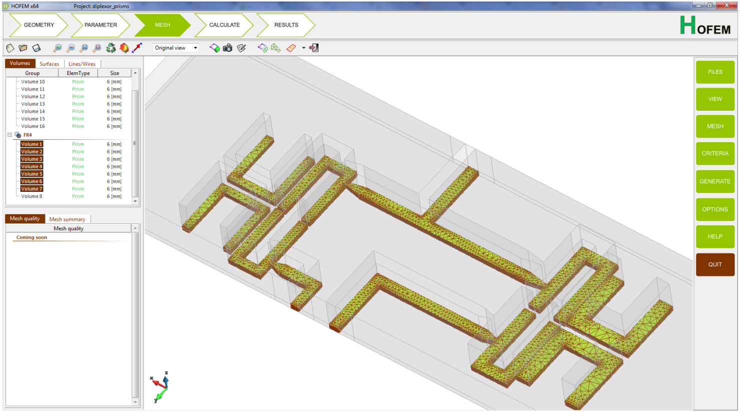
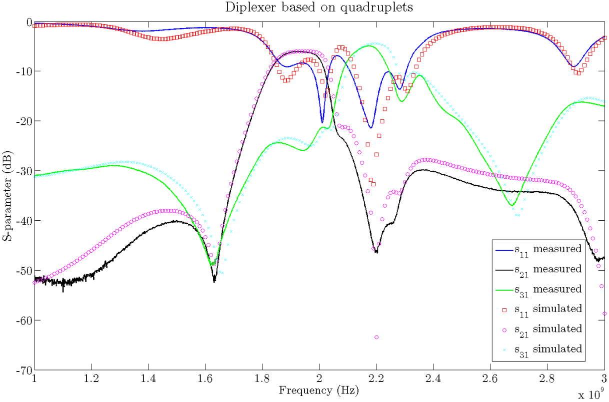


# Comparison (& ii)



# Comparison (& ii)

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# Conclusions

- Systematic approach for designing higher-order basis functions.
- Mathematical verification of the element.
- Competitive with other families of prismatic elements.
- Acknowledgements: TEC2010-18175/TCM and TEC2013-47753-C3-2.
- "Second Order Nedelec Curl-Conforming Prismatic Element for Computational Electromagnetics", *IEEE Transactions on Antennas and Propagation*, submitted Jun 15, advanced state of review.

# Thank you for your attention!

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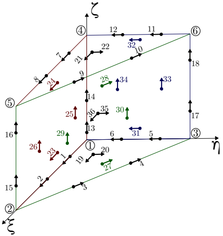



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