On the Design of Higher-Order Curl-Conforming Finite Elements and its Assembly Features

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Universidad Carlos III de Madrid

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18 May 2017

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Motivation

- FEM: domain, basis functions and DOFs.
- Based on:
  - Cartesian coordinates.
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- Known space of functions.
- *A priori* definition of degrees of freedom as functionals.

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Definition of the degrees of freedom

- Edges.

$$g(\mathbf{u}) = \int_e (\mathbf{u} \cdot \hat{\boldsymbol{\tau}}) q \, dl, \forall q \in P_1(e)$$

- Triangular faces.

$$g(\mathbf{u}) = \int_{f_t} (\mathbf{u} \times \hat{\mathbf{n}}) \cdot \mathbf{q} \, ds, \forall \mathbf{q} \in \mathbf{P}_0(f_t)$$

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- Quadrilateral faces.

$$g(\mathbf{u}) = \int_{f_q} (\hat{\mathbf{n}} \times \mathbf{u}) \cdot \mathbf{q} \, ds, \forall \mathbf{q} = (q_1, q_2); q_1 \in \mathcal{Q}_{0,1}; q_2 \in \mathcal{Q}_{1,0}$$

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- Volume.

$$g(\mathbf{u}) = \int_v \mathbf{u} \cdot \mathbf{q} \, dV, \forall \mathbf{q} \in \mathbf{P}_0(f_t)$$

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Definition of the degrees of freedom

Dual basis

$$g_i(\mathbf{N_j}) = \delta_{ij}$$

Use of a master element

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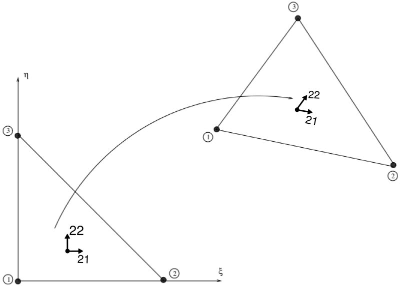

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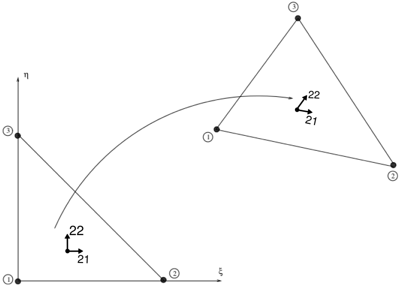

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DOF in faces

- Assembly of edge DOFs.
- Assembly of face DOFs: prism.

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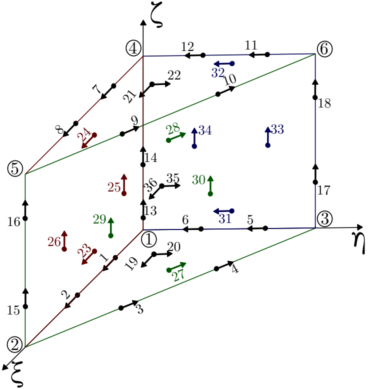

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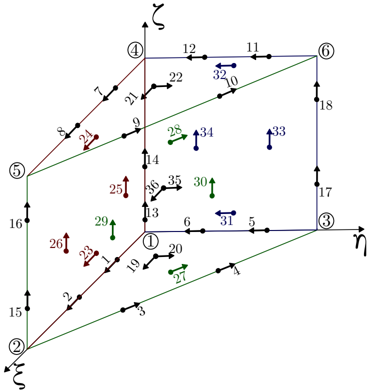
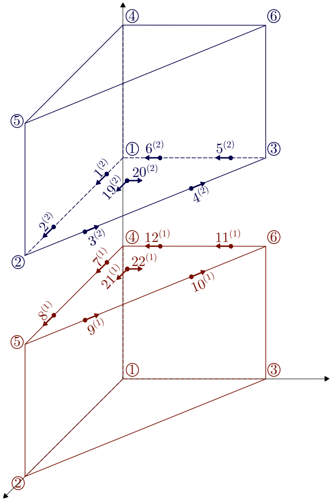
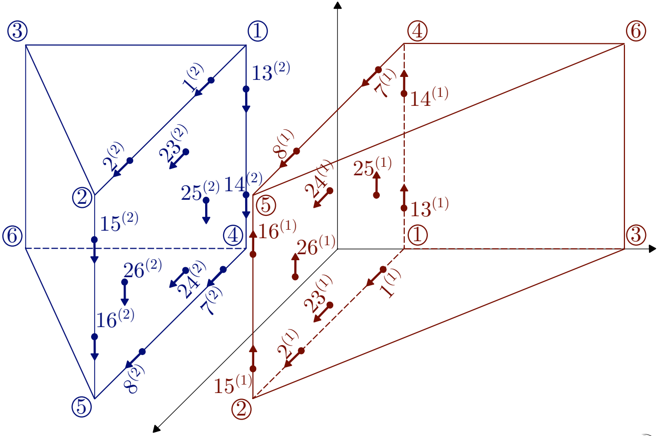


Sharing a triangular face

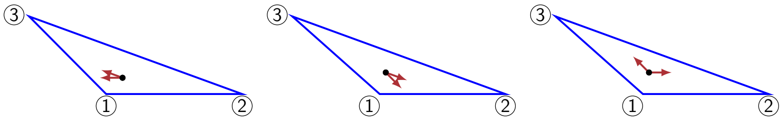


Sharing a quadrilateral face



vc vs vq

- *vc* version.



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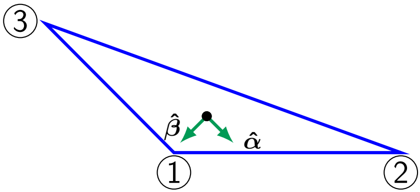

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  - Spectral.

$$
\begin{gathered}
L_m L_l^2 \mathbf{W}_{ij};\ i,j=1,2,3; j>i; m=i,j; l=4,5 \\
L_i^2 L_l \nabla L_l;\ i=1,2,3; l=4,5 \\
L_k L_l^2 \mathbf{W}_{ij};\ i,j,k=1,2,3; j>i; k\neq i,j; l=4,5 \\
L_m L_l L_{l+1} \mathbf{W}_{ij};\ i,j=1,2,3; j>i; m=i,j; l=4 \\
L_i L_j L_l \nabla L_l;\ i,j=1,2,3; j>i; l=4,5 \\
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\tag{1}
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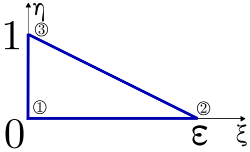
Triangle deformation

$$[M^p] = [D]^{-1}[M][D]^{-1}$$
$$[K^p] = [D]^{-1}[K][D]^{-1}$$
$$D_{ii} = \sqrt{M_{ii}}$$



	Reference prism		Triangle deformation					
			$\varepsilon = 4$		$\varepsilon = 8$		$\varepsilon = 16$	
Version	$[M^p]$	$[K^p]$	$[M^p]$	$[K^p]$	$[M^p]$	$[K^p]$	$[M^p]$	$[K^p]$
vc,(1-2)	81	37	1587	210	18826	791	276385	3096
vc,(2-3)	81	37	217	199	738	733	2827	2856
vc,(3-1)	71	38	215	197	737	732	2825	2854
vq	72	37	215	197	737	732	2826	2854
Graglia	37	19	174	104	639	394	2498	1551
Tobon	171	20	842	101	3468	398	14046	1588

Rectangle deformation

$$[M^p] = [D]^{-1}[M][D]^{-1}$$
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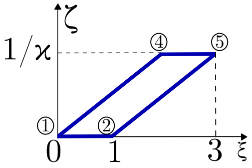
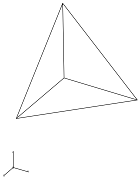
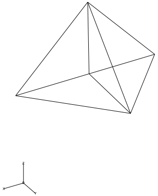
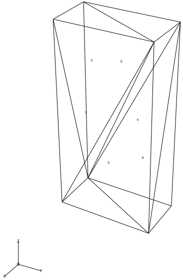


	Reference prism		Rectangle deformation					
			$\kappa = 2$		$\kappa = 4$		$\kappa = 8$	
Version	$[M^p]$	$[K^p]$	$[M^p]$	$[K^p]$	$[M^p]$	$[K^p]$	$[M^p]$	$[K^p]$
vc	72	37	3107	2566	12270	10205	48926	40765
vq	72	37	2187	2066	8435	8171	33432	32599
Graglia	37	19	1484	1067	5889	4279	23509	17131
Tobon	171	20	5967	1209	23559	4226	93928	16923

Tetrahedra







	Parent El.	Example el.2	El. Cube $1 \times 2 \times 4$
vq	128	174	175
vc	138	189	1214

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Thank you for your attention!

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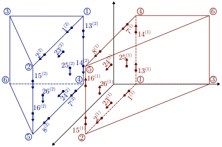

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