H(curl)-Conforming Hierarchical Basis Functions on Prisms and Hexahedra

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Introduction

Hierarchical Incomplete Curl-Conforming Basis Functions on Prisms and Hexahedra

Hierarchical Incomplete Curl-Conforming Basis Functions on Prisms and Hexahedra

$$H(\text{curl}, \Omega) := \left\{ \boldsymbol{w} \in [L_2(\Omega)]^3 \middle| \boldsymbol{\nabla} \times \boldsymbol{w} \in [L_2(\Omega)]^3 \right\}$$

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Hierarchical Incomplete Curl-Conforming Basis Functions on Prisms and Hexahedra

$$H^1(\Omega) := \left\{ v \in [L_2(\Omega)]^3 \middle| \nabla v \in [L_2(\Omega)]^3 \right\}$$

$$\forall\, v_p \in H^1(\Omega), \nabla v_{p+1} \notin \mathbf{w}_p$$

Nédélec, Jean-Claude
Mixed finite elements in R3
Numerische Mathematik, 35(3): 315-341, 1980.

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$$\mathcal{W}_p = \text{span}(\boldsymbol{w}_p)$$

$$\boldsymbol{w}_{p-1} \subset \boldsymbol{w}_p$$

Webb, Jon P.
Hierarchal vector basis functions of arbitrary order for triangular and tetrahedral finite elements
IEEE Transactions on Antennas and Propagation, 47(8): 1244-1253, 1999.

Graglia, Roberto D. and Peterson, Andrew F.
Hierarchical curl-conforming Nédélec elements for quadrilateral and brick cells
IEEE Transactions on Antennas and Propagation, 59(8): 2766-2773, 2011.

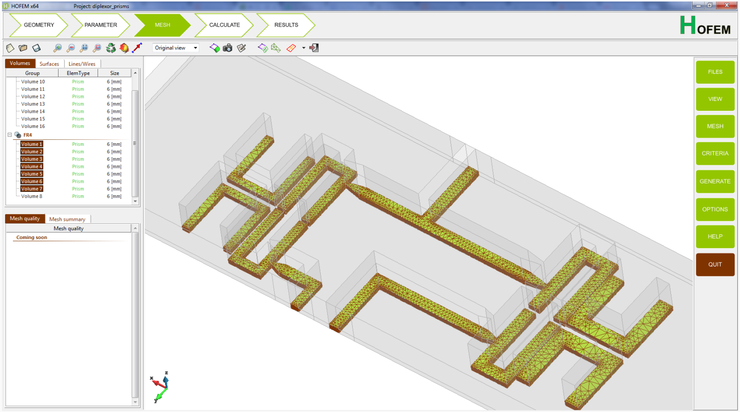
Fuentes, Federico; Keith, Brendan; Demkowicz, Leszek; Nagaraj, Sriram
Orientation embedded high order shape functions for the exact sequence elements of all shapes
Computers & Mathematics with applications, 70(4): 353-458, 2015.

Intro: Proposed bases

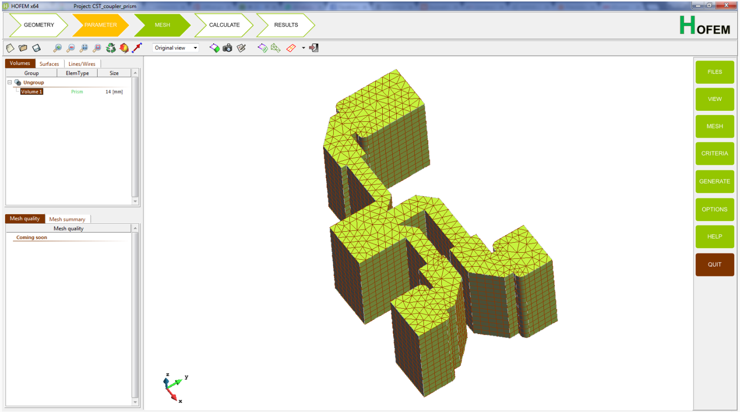
- $\boldsymbol{\pi}_p \boldsymbol{w}_q = 0 \qquad \forall\, \boldsymbol{w}_q \in \tilde{\mathcal{W}}_q,\; q > p$
- Division into gradient and rotational spaces

Ingelström, Pär
A new set of H (curl)-conforming hierarchical basis functions for tetrahedral meshes
IEEE Transactions on Microwave Theory and Techniques, 54(1): 106-114, 2006.

- Planar structures



- Planar structures
- Waveguide sections



Intro: Structured meshes

- Planar structures
- Waveguide sections
- Hybrid meshes

Hierarchical Incomplete Curl-Conforming Basis Functions on
Prisms and Hexahedra

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2. Construction of basis functions
   - Definition of finite element
   - Procedure
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Construction of basis functions

Construction of basis functions

Definition of finite element

- Domain
- Space of functions
- Degrees of freedom

- Affine coordinates

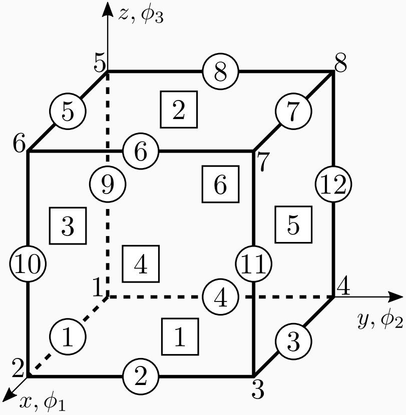
$\phi_i \in [0,1],\ i = 1,\ldots,6$

$$\phi_4 = 1 - \phi_1$$

$$\phi_5 = 1 - \phi_2$$

$$\phi_6 = 1 - \phi_3$$

Fuentes, Federico; Keith, Brendan; Demkowicz, Leszek; Nagaraj, Sriram
Orientation embedded high order shape functions for the exact sequence elements of all shapes
Computers & Mathematics with applications, 70(4): 353-458, 2015.



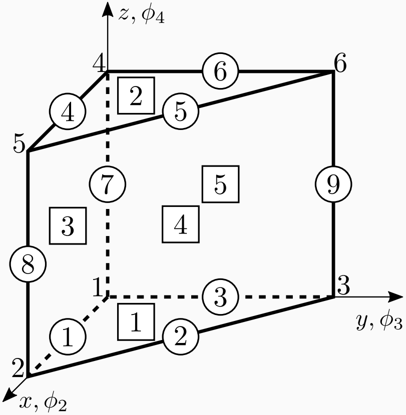
- Affine coordinates

$$\phi_i \in [0,1], \; i = 1, \dots, 5$$

$$\phi_1 = 1 - \phi_2 - \phi_3$$

$$\phi_5 = 1 - \phi_4$$

Fuentes, Federico; Keith, Brendan; Demkowicz, Leszek; Nagaraj, Sriram
Orientation embedded high order shape functions for the exact sequence elements of all shapes
Computers & Mathematics with applications, 70(4): 353-458, 2015.



- Hexahedron

$$
\begin{aligned}
\mathcal{W}_p = & \{\mathbb{P}_{p-1}(I_x) \otimes \mathbb{P}_p(I_y) \otimes \mathbb{P}_p(I_z)\} \times \\
& \{\mathbb{P}_p(I_x) \otimes \mathbb{P}_{p-1}(I_y) \otimes \mathbb{P}_p(I_z)\} \times \\
& \{\mathbb{P}_p(I_x) \otimes \mathbb{P}_p(I_y) \otimes \mathbb{P}_{p-1}(I_z)\}
\end{aligned}
$$

- Triangular prism

$$\mathcal{W}_p = \{\mathcal{R}_p(T_{x,y}) \otimes \mathbb{P}_p(I_z)\} \times \{\mathbb{P}_p(T_{x,y}) \otimes \mathbb{P}_{p-1}(I_z)\}$$

$$\mathcal{R}_p(T_{x,y}) = (\mathbb{P}_{p-1})^2 \oplus \mathcal{S}_p$$

$$\mathcal{S}_p = \{\boldsymbol{w} \in (\tilde{\mathbb{P}}_p)^2 \mid \boldsymbol{w} \cdot \boldsymbol{r} = 0\}$$

- Edges
  $\alpha_p^e(\boldsymbol{w}) = (\boldsymbol{q}^e, \boldsymbol{w})_e = \int_e \boldsymbol{q}^e \cdot \boldsymbol{w}\, de \qquad \forall\, \boldsymbol{q}^e \in \mathbb{P}_{p-1}$

- Edges
- Triangular faces
  $\alpha_p^{\triangle}(\boldsymbol{w}) = (\boldsymbol{q}^{\triangle}, \boldsymbol{w})_f = \int_f \boldsymbol{q}^{\triangle} \cdot \boldsymbol{w} \, df \qquad \forall\, \boldsymbol{q}^{\triangle} \in (\mathbb{P}_{p-2})^2$

- Edges
- Triangular faces
- Rectangular faces

$$\alpha_p^{\square}(\boldsymbol{w}) = (\boldsymbol{q}^{\square}, \boldsymbol{w})_f = \int_f \boldsymbol{q}^{\square} \cdot \boldsymbol{w} \, df \quad \forall\, \boldsymbol{q}^{\square} \in (\{\mathbb{P}_{p-1,p-2}\} \times \{\mathbb{P}_{p-2,p-1}\})$$

- Edges
- Triangular faces
- Rectangular faces
- Volume
  - Hexahedron

$$\alpha_p^v(\boldsymbol{w}) = (\boldsymbol{q}^v, \boldsymbol{w})_v = \int_H \boldsymbol{q}^v \cdot \boldsymbol{w} \, dH$$

$$\forall \, \boldsymbol{q}^v \in (\{\mathbb{P}_{p-1,p-2,p-2}\} \times \{\mathbb{P}_{p-2,p-1,p-2}\} \times \{\mathbb{P}_{p-2,p-2,p-1}\})$$

- Edges
- Triangular faces
- Rectangular faces
- Volume
  - Hexahedron
  - Triangular prism

$$\alpha_p^{\{v_1,v_2\}}(\boldsymbol{w}) = \int_P \boldsymbol{q}^{\{v_1,v_2\}} \cdot \boldsymbol{w}\, dP$$

$$\forall\, \boldsymbol{q}^{v_1} \in (\mathbb{P}_{p-2,p-2,0} \times \mathbb{P}_{p-2,p-2,0} \times \{0\})$$

$$\forall\, \boldsymbol{q}^{v_2} \in (\{0\} \times \{0\} \times \{\mathbb{P}_{p_1,p_2,p-1}\}),\ p_1 + p_2 = p - 3$$

Construction of basis functions

Procedure

1. $\mathcal{V}_p = \tilde{\mathcal{V}}_1 \oplus \cdots \oplus \tilde{\mathcal{V}}_p$ $\qquad \mathcal{V}_p \in H^1(\Omega)$
2. $\mathcal{W}_p = \tilde{\mathcal{W}}_1 \oplus \cdots \oplus \tilde{\mathcal{W}}_p$ $\qquad \mathcal{W}_p \in H(\text{curl}, \Omega)$
3. $\tilde{\mathcal{W}}_1 = \tilde{\mathcal{A}}_1$
4. $\tilde{\mathcal{W}}_p = \tilde{\mathcal{A}}_p \oplus \boldsymbol{\nabla} \tilde{\mathcal{V}}_p$ $\qquad p > 1$
5. Apply $\boldsymbol{\pi}_p \boldsymbol{w}_q = 0$ $\qquad \forall\, \boldsymbol{w}_q \in \tilde{\mathcal{W}}_q,\ q > p$
   - $\alpha_p(\boldsymbol{w} - \boldsymbol{\pi}_p \boldsymbol{w}) = 0$ $\qquad \alpha_p = \alpha_p^e,\ \alpha_p^f,\ \alpha_p^v$
   - $\alpha_{p-1}^e(\tilde{\boldsymbol{w}}_p) = 0$
   - $\alpha_{p-1}^f(\tilde{\boldsymbol{w}}_p) = 0$
   - $\alpha_{p-1}^v(\tilde{\boldsymbol{w}}_p) = 0$

- Whitney functions

$$\boldsymbol{\varpi}(\phi_i, \phi_j) = \boldsymbol{\varpi}_{ij} = \phi_j \boldsymbol{\nabla} \phi_i - \phi_i \boldsymbol{\nabla} \phi_j$$

- Legendre-based polynomials

$$\begin{aligned}
\rho_1(\phi_i, \phi_j) &= \rho_{1,ij} = \phi_i - \phi_j \\
\rho_2(\phi_i, \phi_j) &= \rho_{2,ij} = \phi_i^2 - 3\phi_i\phi_j + \phi_j^2 \\
\kappa_2(\phi_i, \phi_j) &= \kappa_{2,ij} = \phi_i^2 - 4\phi_i\phi_j + \phi_j^2 \\
\rho_3(\phi_i, \phi_j) &= \rho_{3,ij} = \phi_i^3 - 6\phi_i^2\phi_j + 6\phi_i\phi_j^2 - \phi_j^3 \\
\kappa_3(\phi_i, \phi_j) &= \kappa_{3,ij} = \phi_i^3 - 9\phi_i^2\phi_j + 9\phi_i\phi_j^2 - \phi_j^3
\end{aligned}$$

- Useful relations when $\phi_j = 1 - \phi_i$

$$\rho_{1,ij}\boldsymbol{\nabla}\phi_i = \boldsymbol{\nabla}(\phi_i\phi_j)$$
$$\kappa_{2,ij}\boldsymbol{\nabla}\phi_i = \boldsymbol{\nabla}(\phi_i\phi_j\rho_{1,ij}) = \tilde{P}_2(\phi_i)\boldsymbol{\nabla}\phi_i$$
$$\kappa_{3,ij}\boldsymbol{\nabla}\phi_i = \boldsymbol{\nabla}(\phi_i\phi_j\rho_{2,ij}) = \tilde{P}_3(\phi_i)\boldsymbol{\nabla}\phi_i$$

- Scalar expansion factors for $\mathcal{V}_p$

$$\Upsilon_2(\phi_i, \phi_j) = \{\phi_i \phi_j\}$$
$$\Upsilon_3(\phi_i, \phi_j) = \{\Upsilon_2(\phi_i, \phi_j), \phi_i \phi_j \rho_{1,i,j}\}$$
$$\Upsilon_4(\phi_i, \phi_j) = \{\Upsilon_3(\phi_i, \phi_j), \phi_i \phi_j \rho_{2,i,j}\}$$

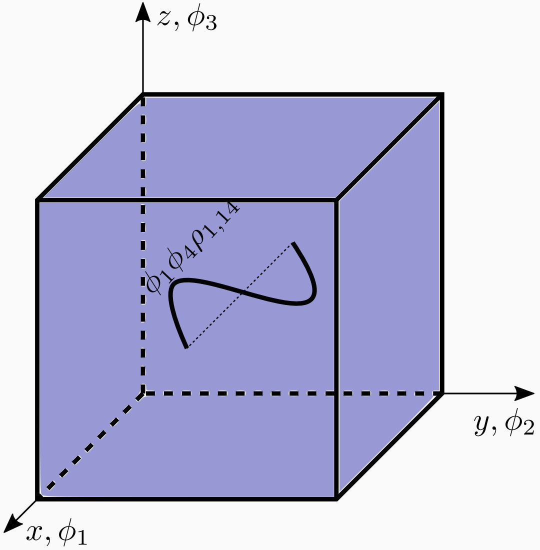
- Vector expansion factors for $\mathcal{A}_p$

$$\boldsymbol{\Xi}'_2(\phi_i, \phi_j) = \boldsymbol{\Xi}_2(\phi_i, \phi_j)$$
$$\boldsymbol{\Xi}'_3(\phi_i, \phi_j) = \{\boldsymbol{\Xi}'_2(\phi_i, \phi_j), \kappa_{2,ij} \boldsymbol{\varpi}_{ij}\}$$
$$\boldsymbol{\Xi}'_4(\phi_i, \phi_j) = \{\boldsymbol{\Xi}'_3(\phi_i, \phi_j), \kappa_{3,ij} \boldsymbol{\varpi}_{ij}\}$$

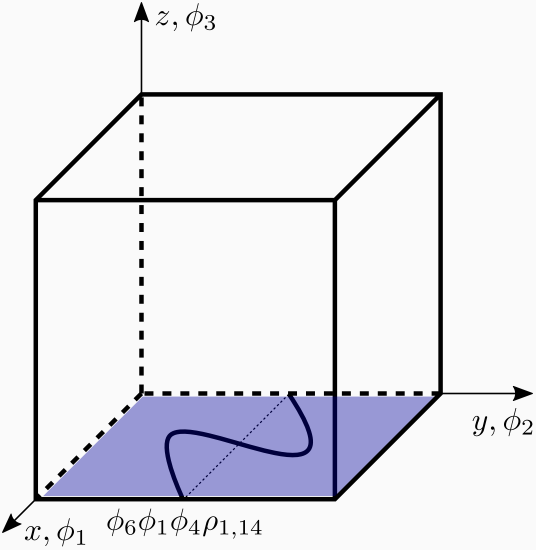
Construction: edge functions

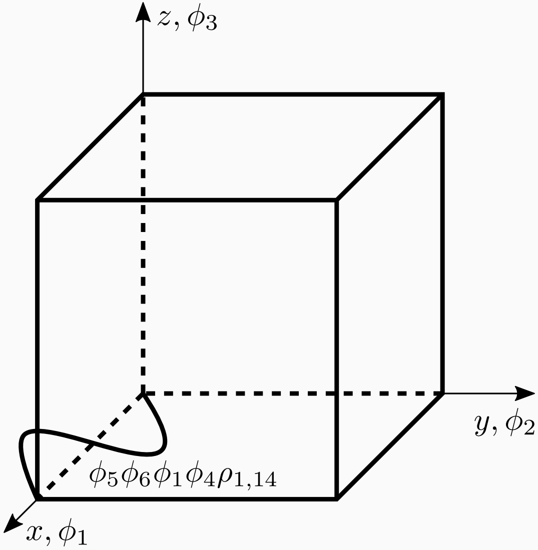
- $\boldsymbol{w}_1 = \phi_k \phi_l \boldsymbol{\varpi}_{ij}$, $\boldsymbol{w}_1 \in \mathcal{A}_1$
- $\boldsymbol{w}_p = \boldsymbol{\nabla} v_p$, $\forall p > 1$

Space	*Basis functions*
$\tilde{\mathcal{V}}_2$	$\phi_k \phi_l \phi_i \phi_j$
$\tilde{\mathcal{V}}_3$	$\phi_k \phi_l \phi_i \phi_j \rho_{1,ij}$
$\tilde{\mathcal{V}}_4$	$\phi_k \phi_l \phi_i \phi_j \rho_{2,ij}$


z, ϕ_3
$\phi_1\phi_4\rho_{1,14}$
y, ϕ_2
x, ϕ_1

Construction: edge functions




z, ϕ_3
y, ϕ_2
$\phi_5\phi_6\phi_1\phi_4\rho_{1,14}$
x, ϕ_1

- Procedure
  1. Build $\boldsymbol{\nabla}\mathcal{V}_p$ enforcing $\boldsymbol{\pi}_p \boldsymbol{\nabla} v_q = 0, \qquad q > p$
  2. Build $\mathcal{W}_p$ with tensor-products, $\boldsymbol{\pi}_p \boldsymbol{w}_q = 0, \qquad q > p$
  3. Move $\dim(\boldsymbol{\nabla}\mathcal{V}_p)$ functions from $\mathcal{W}_p$ to separate spaces
- Example (i): face functions

$$v_p^f = \phi_m \{\Upsilon_p(\phi_i, \phi_j)\} \times \{\Upsilon_p(\phi_k, \phi_l)\}$$

$$\boldsymbol{w}_p^f = \phi_m \left(\{\Upsilon_p(\phi_i, \phi_j)\} \times \{\boldsymbol{\Xi}_p(\phi_k, \phi_l)\} \oplus \{\Upsilon_p(\phi_k, \phi_l)\} \times \{\boldsymbol{\Xi}_p(\phi_i, \phi_j)\} \right)$$

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- Example (ii): second-order face functions

$$\tilde{V}_2^f = \phi_m \phi_i \phi_j \phi_k \phi_l$$

$$\tilde{\boldsymbol{w}}_2^{f,(1)} = \phi_m \phi_i \phi_j \boldsymbol{\varpi}_{kl}$$
$$\tilde{\boldsymbol{w}}_2^{f,(2)} = \phi_m \phi_k \phi_l \boldsymbol{\varpi}_{ij}$$
$$\tilde{\boldsymbol{w}}_2^{f,(3)} = \phi_m \phi_i \phi_j \rho_{1,kl} \boldsymbol{\varpi}_{kl}$$
$$\tilde{\boldsymbol{w}}_2^{f,(4)} = \phi_m \phi_k \phi_l \rho_{1,ij} \boldsymbol{\varpi}_{ij}$$

$$\nabla \tilde{V}_2^f = \tilde{\boldsymbol{w}}_2^{f,(3)} + \tilde{\boldsymbol{w}}_2^{f,(4)}$$

Numerical results

Numerical results

Single element

Results: condition number

$$M_{ij} = (\boldsymbol{w}_i, \boldsymbol{w}_j)_v$$

$$\boldsymbol{M}_p = \boldsymbol{D}_M^{-1} \boldsymbol{M} \boldsymbol{D}_M^{-1}, \ D_{M,ii} = \sqrt{M_{ii}}$$

$$K_{ij} = (\boldsymbol{\nabla} \times \boldsymbol{w}_i, \boldsymbol{\nabla} \times \boldsymbol{w}_j)_v$$

$$\boldsymbol{K}_p = \boldsymbol{D}_K^{-1} \boldsymbol{M} \boldsymbol{D}_K^{-1}, \ D_{K,ii} = \sqrt{K_{ii}}$$

Prism	$\boldsymbol{M}_p$		$\boldsymbol{K}_p$	
	Fuentes, [1]	Proposed	Fuentes, [1]	Proposed
Order 2	730	1081	34	74
Order 3	3193	2095	71	91
Order 4	40780	11380	198	182

$$M_{ij} = (\boldsymbol{w}_i, \boldsymbol{w}_j)_v$$

$$\boldsymbol{M}_p = \boldsymbol{D}_M^{-1} \boldsymbol{M} \boldsymbol{D}_M^{-1},\ D_{M,ii} = \sqrt{M_{ii}}$$

$$K_{ij} = (\boldsymbol{\nabla} \times \boldsymbol{w}_i, \boldsymbol{\nabla} \times \boldsymbol{w}_j)_v$$

$$\boldsymbol{K}_p = \boldsymbol{D}_K^{-1} \boldsymbol{M} \boldsymbol{D}_K^{-1},\ D_{K,ii} = \sqrt{K_{ii}}$$

Hexahedra	$\boldsymbol{M}_p$		$\boldsymbol{K}_p$	
	Fuentes, [1]	Proposed	Fuentes, [1]	Proposed
Order 2	527	2069	25	65
Order 3	527	2540	28	81
Order 4	4687	23183	92	214

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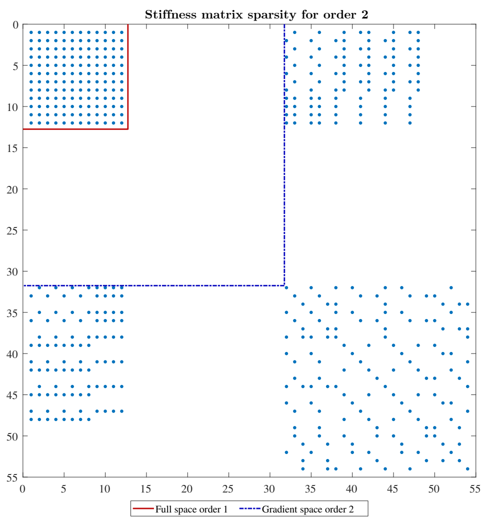
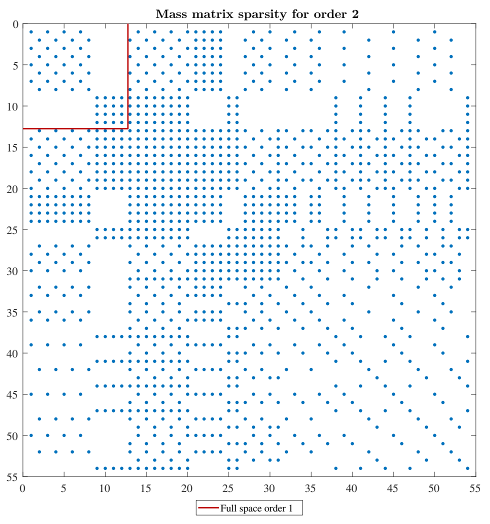
	M_p		
	Order 2	Order 3	Order 4
Proposed	2069	2540	23183
Without division in interior bases	1515	1835	12443
Without division in interior and face bases	687	1517	9898
Without any division	527	783	9191

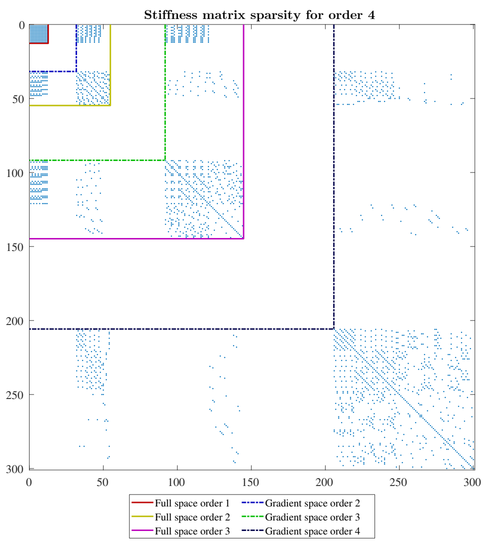
Results: sparsity patterns



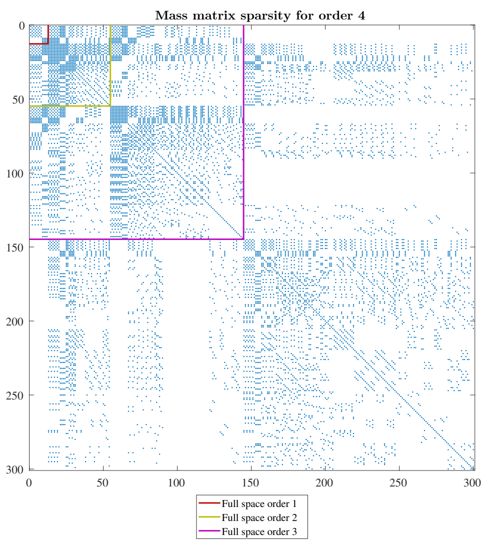
Results: sparsity patterns



Results: sparsity patterns



Results: sparsity patterns



Numerical results

Convergence

- Formulation

$$\nabla \times \mu_r^{-1} \nabla \times E - k_0^2 \varepsilon_r E = 0$$

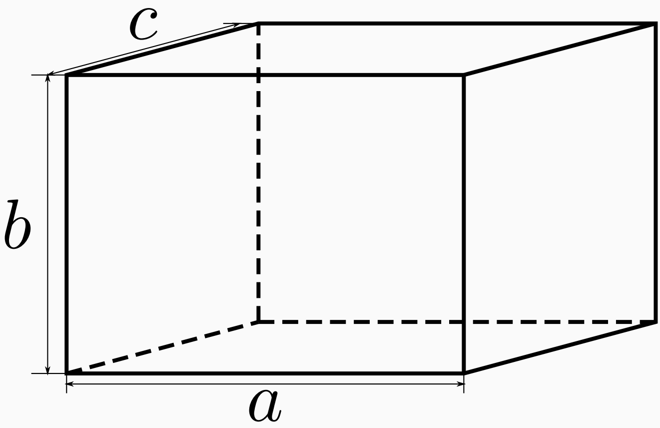
$$\hat{n} \times E = 0 \quad \text{on } \partial\Omega$$

$$(\nabla \times w, \mu_r^{-1} \nabla \times E)_\Omega - k_0^2 (w, \varepsilon_r E)_\Omega = 0$$

$$(K - k_0^2 M) v = 0$$

- Relative error

$$\vartheta = \frac{k_{0,\mathrm{anal}}^2 - k_{0,\mathrm{FEM}}^2}{k_{0,\mathrm{anal}}^2},$$


c
b
a

Results: convergence rates (& ii)

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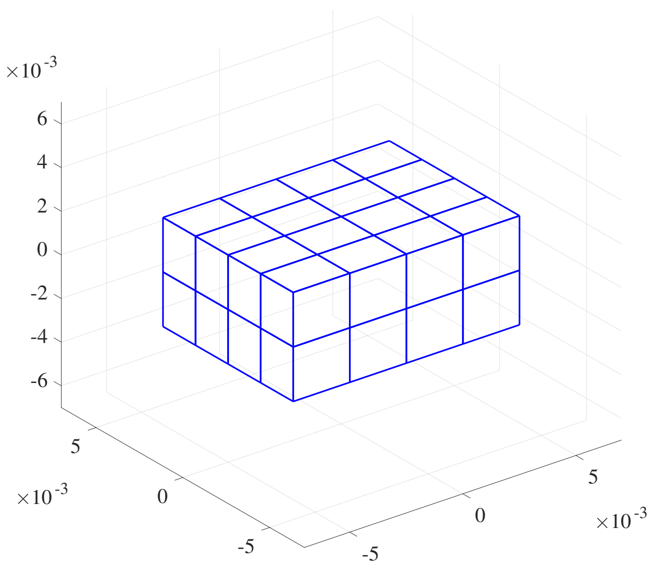
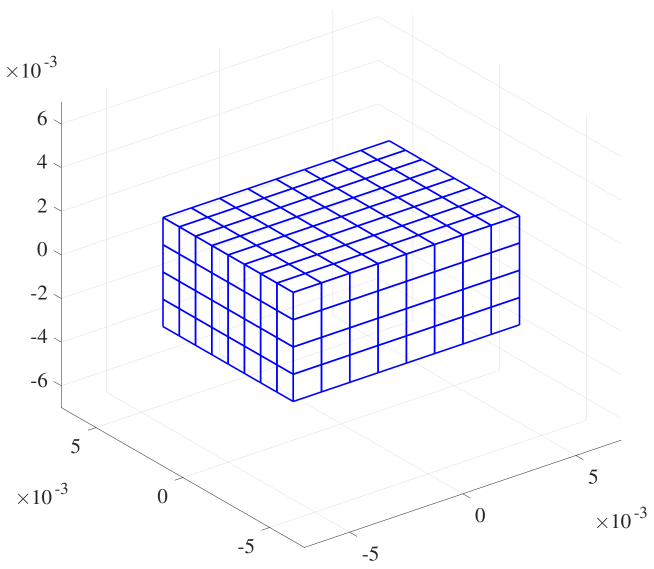
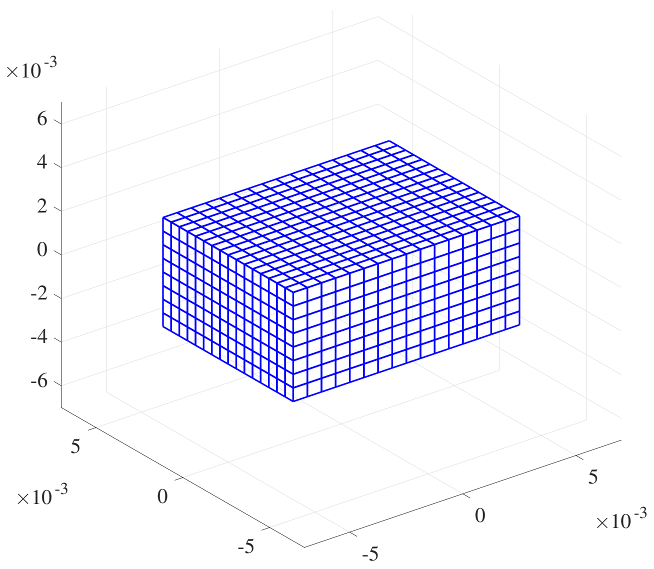
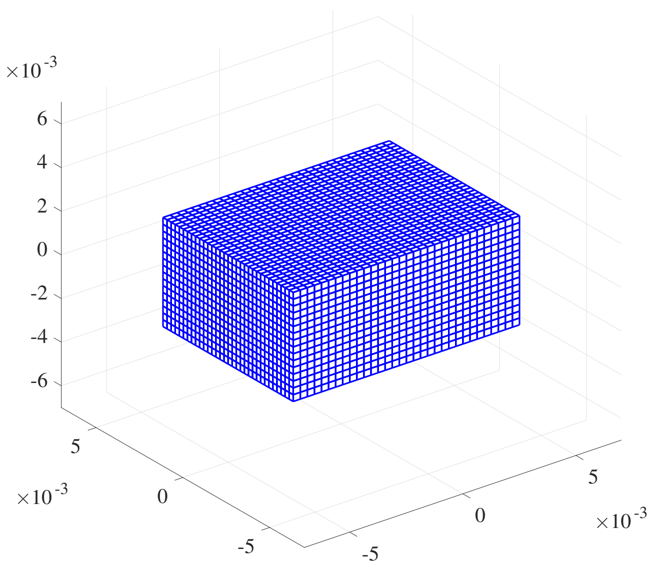


Results: convergence rates (& ii)




×10^-3
6
4
2
0
-2
-4
-6
5
×10^-3
0
-5
5
0
×10^-3
-5


×10^-3
6
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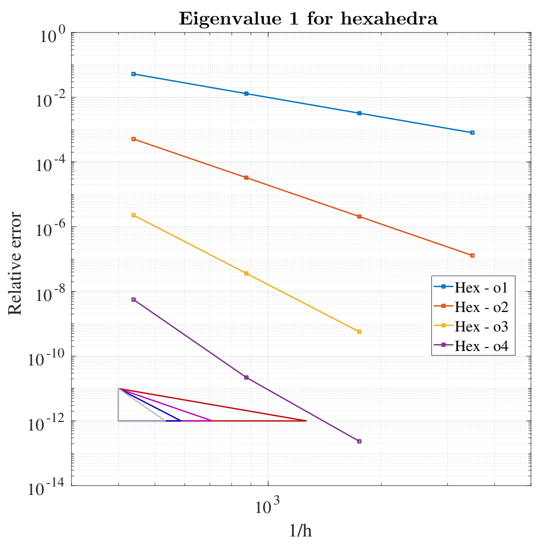
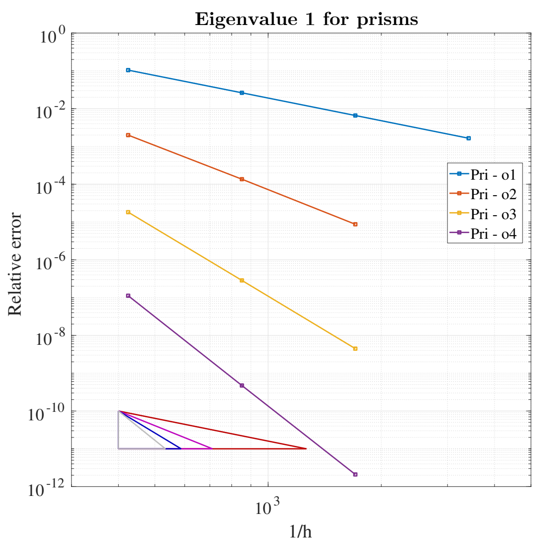
×10^-3
0
-5
-5
0
5
×10^-3



	Slope
Order 1	2.003
Order 2	4.002
Order 3	6.004
Order 4	8.019



	Slope
Order 1	1.996
Order 2	3.955
Order 3	5.985
Order 4	7.802

Conclusions

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- New set of basis functions for structured meshes
  - Orthogonality through interpolation operator
  - Division between gradient and rotational spaces
- Condition number in progress for hexahedra
- Validated with electromagnetic cavities

- Complete space of basis functions
- Concerns with the assembly in hexahedra
- Generic procedure for obtaining interior functions
- Universal matrices with hierarchical scalar bases
- Compatibility of the whole family

Thanks for your attention!

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