A Priori Verification Method for Curl-Conforming Vector Functions in Simplices

Adrian Amor-Martin[1] Luis E. Garcia-Castillo[1]

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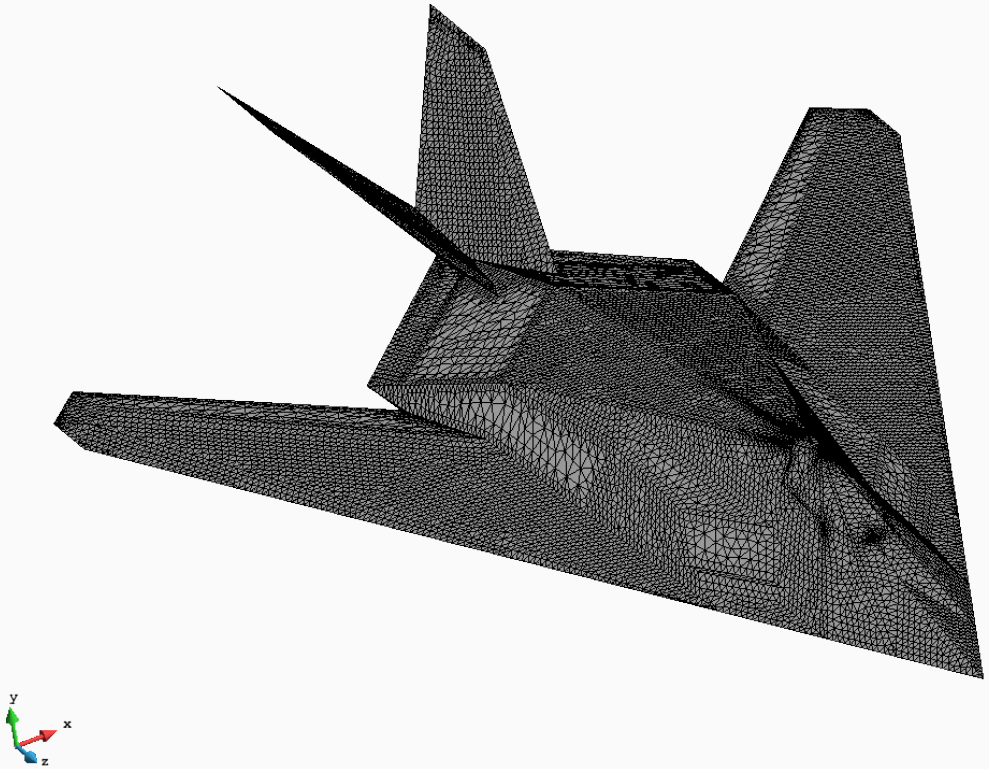
[1] Radiofrequency, Microwaves, Electromagnetics and Antennas Group.
University Carlos III of Madrid.
[aamor,legcasti]@ing.uc3m.es

Introduction

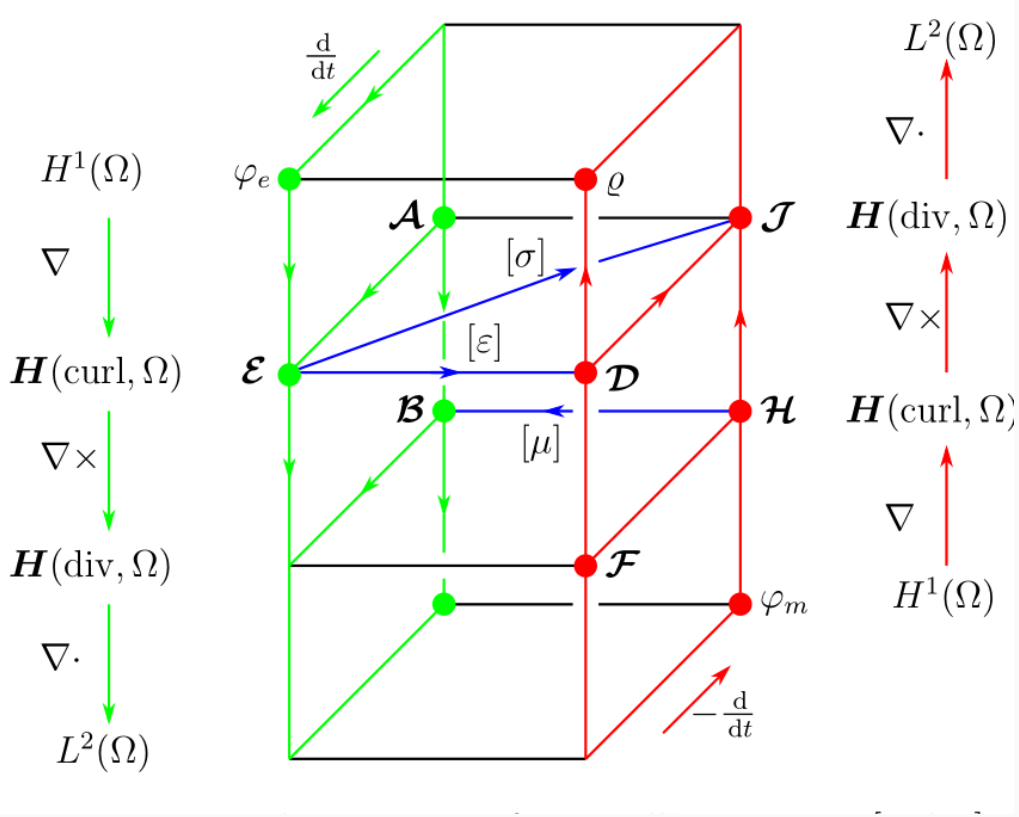
A Priori Verification Method for Curl-Conforming Vector Functions in Simplices



A Priori Verification Method for Curl-Conforming Vector Functions in Simplices

$$H(\mathrm{curl}, \Omega) := \left\{ \boldsymbol{w} \in [L_2(\Omega)]^3 \middle| \boldsymbol{\nabla} \times \boldsymbol{w} \in [L_2(\Omega)]^3 \right\}$$

A Priori Verification Method for Curl-Conforming Vector Functions in Simplices



R. Baltes, Verfahren der Ordnungsreduktion zur Charakterisierung Passiver Mikrowellenstrukturen im Frequenz- und Zeitbereich auf Basis der Finite-Elemente-Methode, Ph.D. thesis, 2018

A Priori Verification Method for Curl-Conforming Vector Functions in Simplices

$$H^1(\Omega) := \left\{ v \in [L_2(\Omega)]^3 \middle| \nabla v \in [L_2(\Omega)]^3 \right\}$$

$$\forall\, v_p \in H^1(\Omega), \nabla v_{p+1} \notin \boldsymbol{w}_p$$

Nédélec, Jean-Claude
Mixed finite elements in R3
Numerische Mathematik, 35(3): 315-341, 1980.

A Priori Verification Method for Curl-Conforming Vector Functions in Simplices

- Same rate of convergence for the energy error.
- The reduction only affects the null space of the curl operator.
- Mixed-order discretization of the space

	Mixed-order space	*Full-order space*
Tetrahedron	$\frac{1}{2}p(p+2)(p+3)$	$\frac{1}{2}(p+1)(p+2)(p+3)$
Edge	$6p$	$6(p+1)$
Face	$4p(p-1)$	$4(p+1)(p-1)$
Interior	$\frac{1}{2}p(p-1)(p-2)$	$\frac{1}{2}(p+1)(p-1)(p-2)$

Intro: state of the art (i)

Webb, Jon P.
Hierarchal vector basis functions of arbitrary order for triangular and tetrahedral finite elements
IEEE Transactions on Antennas and Propagation, 47(8): 1244-1253, 1999.

Bluck, M. J.
Conforming Hierarchical Basis Functions
Communications in Computational Physics, 12(4): 1215-1256, 2012.

Beuchler, Sven, Pillwein, Veronika and Zaglmayr, Sabine
Sparsity Optimized High Order Finite Element Functions for H(Curl) on Tetrahedra
Advances in Applied Mathematics, 50(5): 749-769, 2013.

Intro: state of the art (ii)

Graglia, Roberto D., Peterson, Andrew F., and Andriulli, Francesco P.
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A new set of H(curl)-conforming hierarchical basis functions for tetrahedral meshes
IEEE Transactions on Microwave Theory and Techniques, 54(1): 106-114, 2006.

A Priori Verification Method for Curl-Conforming Vector Functions in Simplices

Prove that $\mathcal{W}_p = \text{span}(\boldsymbol{w}_p)$

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Typical verification of basis functions

Typical verification of basis functions

Mixed-order space of functions

Mixed-order space of functions

$$\mathcal{R}_k = \left\{ u \in P_k ; \epsilon^k(u) = 0 \right\}$$

$$P_{k-1} \subset \mathcal{R}_k \subset P_k$$

$$\mathcal{R}_k = P_{k-1} \oplus \mathcal{S}_k$$

$$u \in \mathcal{S}_k \Leftrightarrow r \cdot u = 0$$

Nédélec, Jean-Claude
Mixed finite elements in R3
Numerische Mathematik, 35(3): 315-341, 1980.

First-order basis functions

$$\boldsymbol{w}_i = \begin{Bmatrix} a_1^{(i)} \\ b_1^{(i)} \\ c_1^{(i)} \end{Bmatrix} + \begin{Bmatrix} D^{(i)}\eta - F^{(i)}\xi\zeta \\ -D^{(i)}\xi - E^{(i)}\zeta \\ E^{(i)}\eta + F^{(i)}\xi \end{Bmatrix}$$

Second-order basis functions

$$\boldsymbol{w}_i = \left\{ \begin{array}{c} a_1^{(i)} + a_2^{(i)}\xi + a_3^{(i)}\eta + a_4^{(i)}\zeta \\ \\ b_1^{(i)} + b_2^{(i)}\xi + b_3^{(i)}\eta + b_4^{(i)}\zeta \\ \\ c_1^{(i)} + c_2^{(i)}\xi + c_3^{(i)}\eta + c_4^{(i)}\zeta \end{array} \right\} + \ldots$$

$$\left\{ \begin{array}{c} D^{(i)}\eta^2 - F^{(i)}\xi\eta - G^{(i)}\xi\zeta + H^{(i)}\zeta^2 + J^{(i)}\eta\zeta \\ \\ -D^{(i)}\xi\eta - E^{(i)}\eta\zeta + F^{(i)}\xi^2 + I^{(i)}\zeta^2 - J^{(i)}\xi\zeta + K^{(i)}\xi\zeta \\ \\ E^{(i)}\eta^2 + G^{(i)}\xi^2 - H^{(i)}\xi\zeta - I^{(i)}\eta\zeta - K^{(i)}\xi\eta \end{array} \right\}$$

Typical verification of basis functions

Typical verification methods

Involved matrices

- Mass matrix M

$$M_{ij} = \int_{\Omega} \boldsymbol{w}_i \cdot \boldsymbol{w}_j d\Omega$$

- Stiffness matrix K

$$K_{ij} = \int_{\Omega} (\boldsymbol{\nabla} \times \boldsymbol{w}_i) \cdot (\boldsymbol{\nabla} \times \boldsymbol{w}_j) d\Omega$$

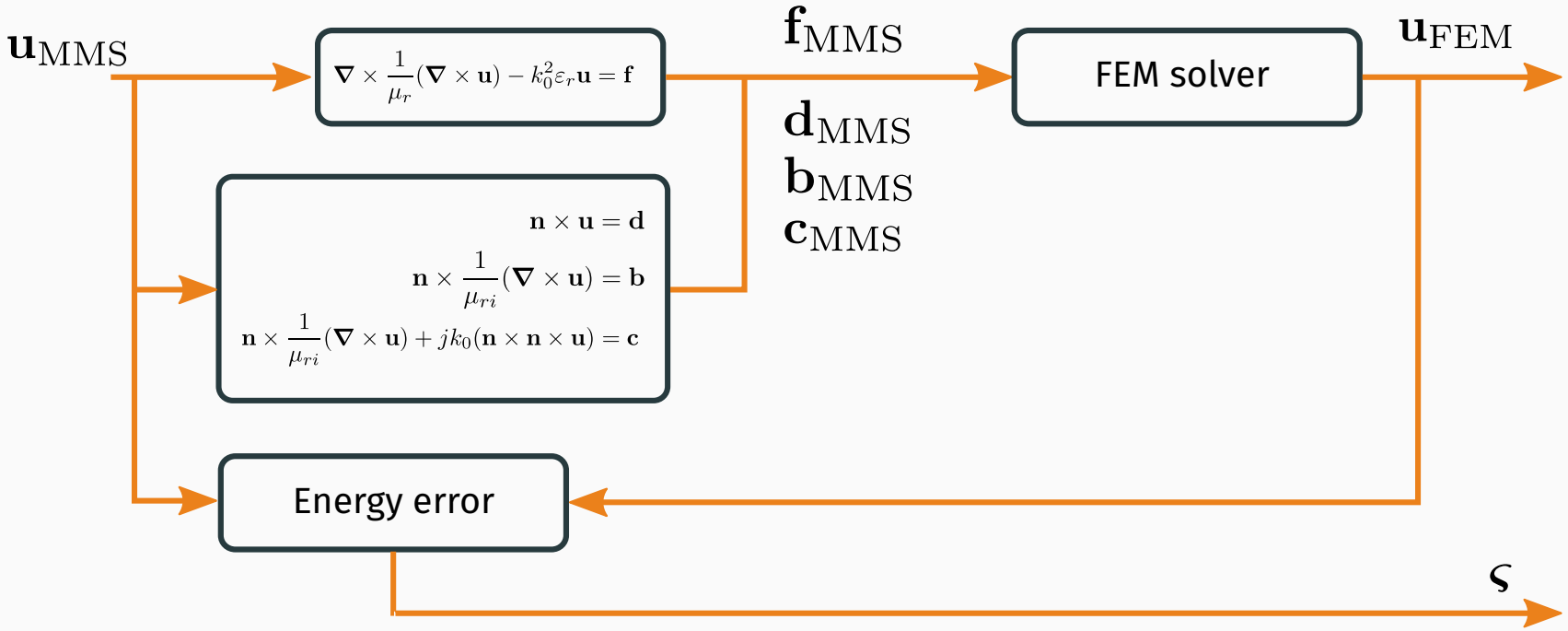
Sanity checks

- M is positive-definite.
- K is positive semi-definite
  - $\#(\lambda = 0) = \text{size}\left(H_k^1(\Omega)\right) - 1$

Extended matrix

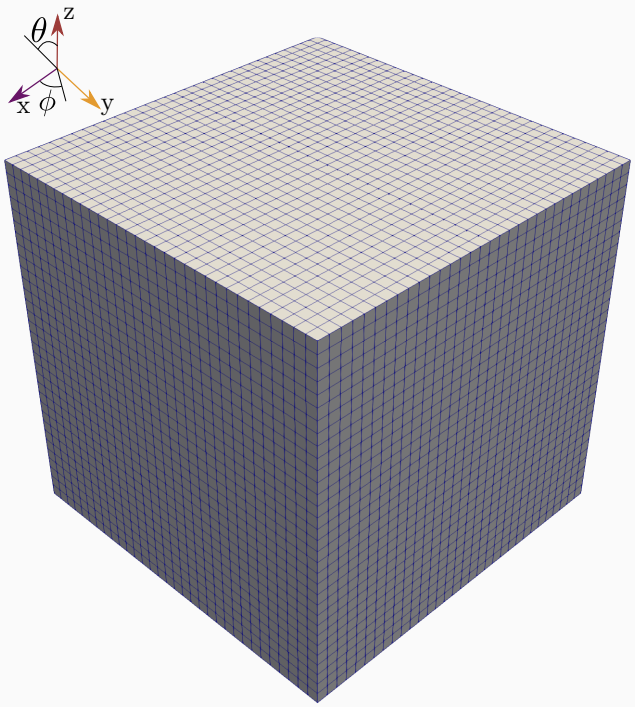
$$M_{\text{extended}} = \begin{bmatrix} M_{\text{polynomials}} & M_{\text{polynomials-bases}} \\ M_{\text{bases-polynomials}} & M_{\text{bases}} \end{bmatrix}$$

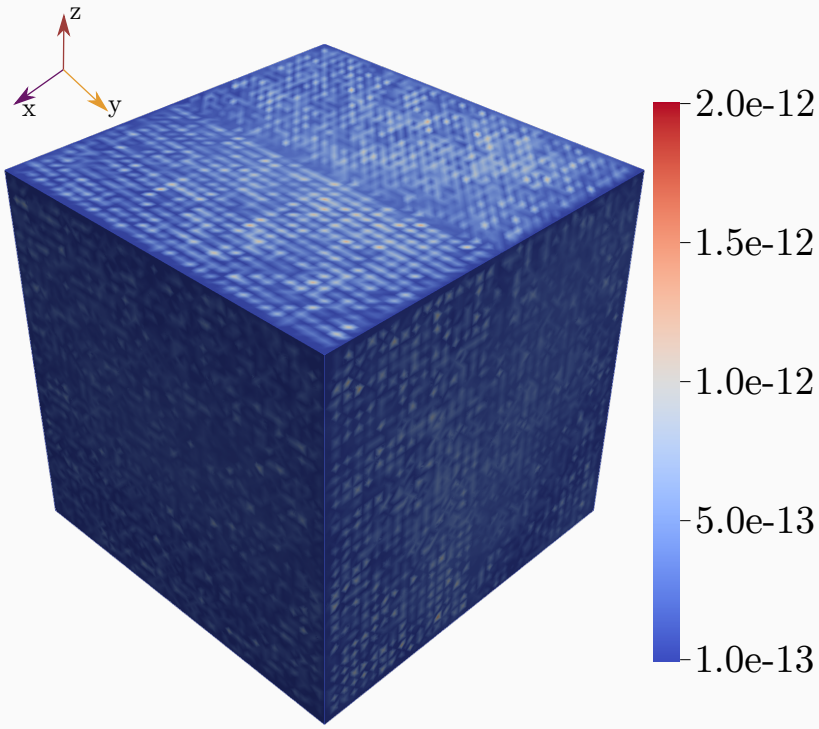
Method of Manufactured Solutions



Garcia-Doñoro, D., Garcia-Castillo, L. E., and Ting, S. W.
Verification Process of Finite-Element Method Code for Electromagnetics.
IEEE Antennas and Propagation Magazine, 1045(9243/16), 2016.

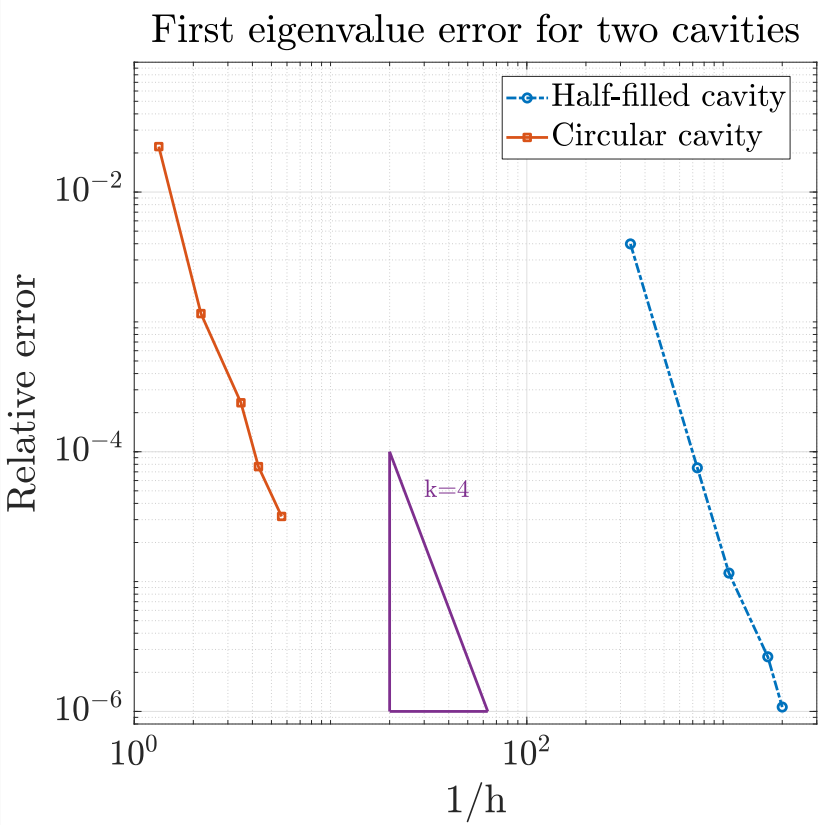
Test with polynomials within $\mathcal{R}_k$





Amor-Martin, A., García-Castillo, L.E.
Second-Order Nédélec Curl-Conforming Hexahedral Element for Computational Electromagnetics.
IEEE Transactions on Antennas and Propagation, 71(1), 859-868, 2023.

Validation with analytical solutions



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A priori verification method in simplices

Motivation

- The former methods are not typical in all the communities.
- All of them need an FEM framework!

Objective revisited

A Priori Verification Method for Curl-Conforming Vector Functions in Simplices

Prove that $\mathcal{W}_p = \text{span}(\boldsymbol{w}_p)$ without using an FEM framework.

Alternative definition based on multi-indices

$$\mathcal{I}(N,k) = \left\{ \boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \ldots, \alpha_N) : \alpha_i \geq 0, \alpha_i \in \mathbb{Z}, \sum_{i=1}^{N} \alpha_i = k \right\}$$

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$$

$$\boldsymbol{e}_l \to \boldsymbol{e}_1 = (1, 0, \ldots, 0)$$

$$\alpha_i < 0 \to \alpha_i = 0$$

$$|\boldsymbol{\alpha}| = \sum_{i=1}^{N} \alpha_i$$

Example with polynomial in $\mathcal{S}_k$ (i)

$$\boldsymbol{w}_i = \left\{ \begin{array}{c} a_1^{(i)} + a_2^{(i)}\xi + a_3^{(i)}\eta + a_4^{(i)}\zeta \\ \\ b_1^{(i)} + b_2^{(i)}\xi + b_3^{(i)}\eta + b_4^{(i)}\zeta \\ \\ c_1^{(i)} + c_2^{(i)}\xi + c_3^{(i)}\eta + c_4^{(i)}\zeta \end{array} \right\} + \ldots$$

$$\left\{ \begin{array}{c} D^{(i)}\eta^2 - F^{(i)}\xi\eta - G^{(i)}\xi\zeta + H^{(i)}\zeta^2 + J^{(i)}\eta\zeta \\ \\ -D^{(i)}\xi\eta - E^{(i)}\eta\zeta + F^{(i)}\xi^2 + I^{(i)}\zeta^2 - J^{(i)}\xi\zeta + K^{(i)}\xi\zeta \\ \\ E^{(i)}\eta^2 + G^{(i)}\xi^2 - H^{(i)}\xi\zeta - I^{(i)}\eta\zeta - K^{(i)}\xi\eta \end{array} \right\}$$

Example with polynomial in $\mathcal{S}_k$ (i)

- $D^{(i)}$ is associated to polynomial $\boldsymbol{s}_D \in \mathcal{S}_2$,

$$\boldsymbol{s}_D = \begin{Bmatrix} \eta^2 \\ -\xi\eta \\ 0 \end{Bmatrix} \rightarrow \boldsymbol{s}_D = \begin{Bmatrix} x_2^2 \\ -x_1 x_2 \\ 0 \end{Bmatrix} \rightarrow \boldsymbol{s}_D = \begin{Bmatrix} \boldsymbol{x}^{\boldsymbol{\alpha}_1}, \boldsymbol{\alpha}_1 = [0,2,0] \\ -\boldsymbol{x}^{\boldsymbol{\alpha}_2}, \boldsymbol{\alpha}_2 = [1,1,0] \\ 0 \end{Bmatrix}$$

Third characterization of the Nédélec space

- $s \in \mathcal{S}_k$ if and only if

$$s = \sum_{l=1}^{N} \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha},l} \mathbf{x}^{\boldsymbol{\alpha}} \boldsymbol{e}_l \text{ with } \sum_{l=1}^{N} c_{\boldsymbol{\beta}-\mathbf{e}_l,l} = 0 \; \forall \; |\boldsymbol{\beta}| = k+1$$

Gopalakrishnan, J., García-Castillo, L.E., Demkowicz, L.F.
Nédélec spaces in affine coordinates.
Computers & Mathematics with Applications, 49(7), 1285-1294, 2005.

Third characterization of the Nédélec space

- $\boldsymbol{s} \in \boldsymbol{\mathcal{S}}_k$ if and only if

$$\boldsymbol{s} = \sum_{l=1}^{N} \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha},l} \mathbf{x}^{\boldsymbol{\alpha}} \boldsymbol{e}_l \text{ with } \sum_{l=1}^{N} c_{\boldsymbol{\beta}-\mathbf{e}_l,l} = 0 \ \forall \ |\boldsymbol{\beta}| = k+1$$

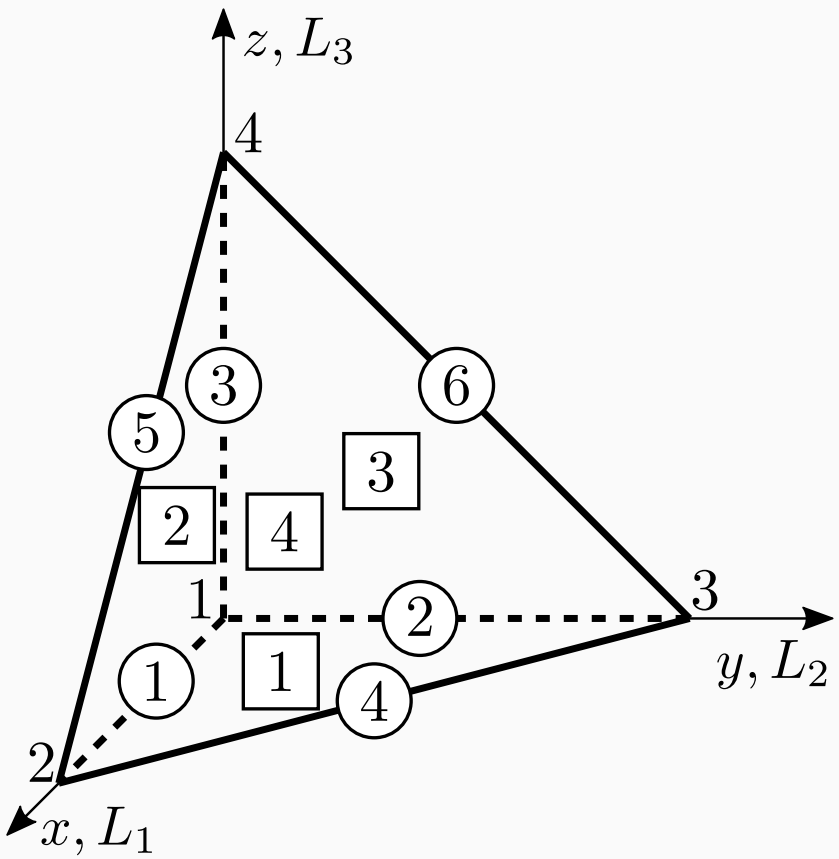
$$\boldsymbol{s}_D = \begin{Bmatrix} x_2^2 \\ -x_1 x_2 \\ 0 \end{Bmatrix} \rightarrow \boldsymbol{s}_D = \begin{Bmatrix} \mathbf{x}^{\boldsymbol{\beta}_1-\mathbf{e}_1}, \boldsymbol{\beta}_1 = [1,2,0] \\ -\mathbf{x}^{\boldsymbol{\beta}_2-\mathbf{e}_2}, \boldsymbol{\beta}_2 = [1,2,0] \\ 0 \end{Bmatrix}$$

$$\boldsymbol{s}_D = \mathbf{x}^{\boldsymbol{\beta}_1-\mathbf{e}_1}\mathbf{e}_1 - \mathbf{x}^{\boldsymbol{\beta}_2-\mathbf{e}_2}\mathbf{e}_2$$

- Clearly, $c_{\boldsymbol{\beta}_1-\mathbf{e}_1,1} = 1$, and $c_{\boldsymbol{\beta}_2-\mathbf{e}_2,2} = -1$, so $\sum_{l=1}^{N} c_{\boldsymbol{\beta}-\mathbf{e}_l,l} = 0$ holds $\forall \ |\boldsymbol{\beta}| = 2+1$, thus $\boldsymbol{s}_D \in \boldsymbol{\mathcal{S}}_2$.
- Note that $\boldsymbol{\alpha}_1 = \boldsymbol{\beta}_1 - \mathbf{e}_1$, $\boldsymbol{\alpha}_2 = \boldsymbol{\beta}_2 - \mathbf{e}_2$.

Affine coordinates for simplices

$$L_4 = 1 - L_1 - L_2 - L_3$$



Third characterization for affine coordinates

$$\begin{tikzcd}
\underline{P^{N+1}_{k+1}} \arrow[r, "\nabla"] \arrow[d, "A_0"'] & \underline{\boldsymbol{P}^{N+1}_{k}} \arrow[d, "\boldsymbol{A}"'] \\
P^{N}_{k+1} \arrow[r, "\nabla"] & \boldsymbol{P}^{N}_{k}
\end{tikzcd}$$

1. Substitute L_i for x_i
2. Substitute $\boldsymbol{\nabla} L_i$ for $\mathbf{e}_i$

Gopalakrishnan, J., García-Castillo, L.E., Demkowicz, L.F.
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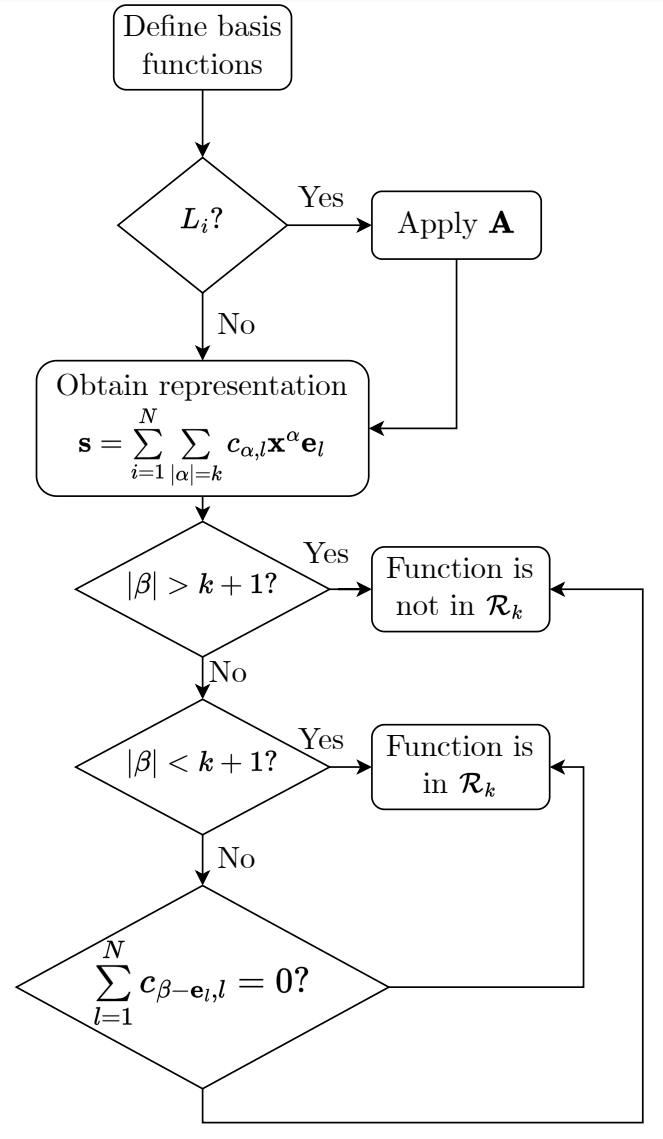
Computers & Mathematics with Applications, 49(7), 1285-1294, 2005.

Flowchart

- Symbolic toolbox from Matlab.
- Input: definition of basis functions.



Results

Results (i)

Webb, Jon P.
Hierarchal vector basis functions of arbitrary order for triangular and tetrahedral finite elements
IEEE Transactions on Antennas and Propagation, 47(8): 1244-1253, 1999.

- Pioneering work in hierarchical basis functions.
- $\mathcal{R}_{30}$ face function not in $\mathcal{R}_3$
- $\mathcal{R}_{300}$ non-orthogonalized volume function not in $\mathcal{R}_3$
- $\mathcal{R}_{300}$, $\mathcal{R}_{301}$, and $\mathcal{R}_{302}$ non-orthogonalized functions not in $\mathcal{R}_3$

Results (ii)

Beuchler, Sven, Pillwein, Veronika and Zaglmayr, Sabine
Sparsity Optimized High Order Finite Element Functions for H(Curl) on Tetrahedra
Advances in Applied Mathematics, 50(5): 749-769, 2013.

- Full-order space of basis functions.
- Face functions $\phi_{ij}^{F,1}$ not in the space of functions.

Results (iii)

Bluck, M. J.
Conforming Hierarchical Basis Functions
Communications in Computational Physics, 12(4): 1215-1256, 2012.

- Unified process for div- and curl-conforming functions for all the shapes.
- Face functions: coefs. for $\boldsymbol{\alpha} = (1, 1, 2, 0)$ sum $-4.56 \cdot 10^{-7}$
- Face functions: coefs. for $\boldsymbol{\alpha} = (2, 1, 1, 0)$ sum $-2.15 \cdot 10^{-4}$
- Face functions: coefs. for $\boldsymbol{\alpha} = (1, 1, 2, 0)$ sum $-1.11 \cdot 10^{-3}$

Results (iv)

Graglia, Roberto D., Peterson, Andrew F., and Andriulli, Francesco P.
Hierarchical curl-conforming Nédélec elements for Triangles and Tetrahedra
IEEE Transactions on Antennas and Propagation, 59(3): 950-959, 2011.

- Hierarchical basis functions with ad-hoc hierarchical polynomials over Whitney forms to minimize condition number.
- All functions within $\mathcal{R}_k$.

Results (v)

Ingelström, Pär
A new set of H(curl)-conforming hierarchical basis functions for tetrahedral meshes
IEEE Transactions on Microwave Theory and Techniques, 54(1): 106-114, 2006.

- Hierarchical basis functions with division within range and nullspace.
- All functions within $\mathcal{R}_k$.

Results (and vi)

Fuentes, Federico; Keith, Brendan; Demkowicz, Leszek; Nagaraj, Sriram
Orientation embedded high order shape functions for the exact sequence elements of all shapes
Computers & Mathematics with applications, 70(4): 353-458, 2015.

- Hierarchical basis functions with orthogonal polynomials over Whitney functions.
- All functions within $\mathcal{R}_k$.

Conclusions

Conclusions

- A priori verification method for $\boldsymbol{\mathcal{R}}_k$ without an FEM framework.
- Development of a Matlab-based tool.
- Some published works are out of $\boldsymbol{\mathcal{R}}_k$.

Thanks for your attention!

Adrian Amor-Martin
aamor@ing.uc3m.es
aamorm.github.io