Adrián Amor Martín, Luis E. García-Castillo, Sergio Llorente-Romano
[aamor,legcasti,sllorent]@ing.uc3m.es
aamorm.github.io
Radiofrequency, Microwaves, Electromagnetics and Antennas Group
Universidad Carlos III de Madrid

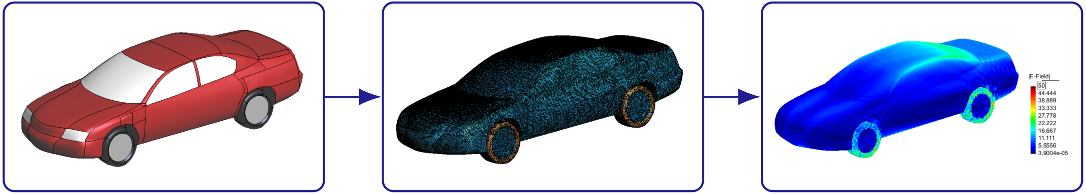
Jul 7, 2025

Anisotropic Nédélec Curl-Conforming Prismatic Element

CMMSE 2025

Introduction

Anisotropic Nédélec Curl-Conforming Prismatic Element



$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) - \omega^2 \mu \varepsilon \mathbf{E} = \mathbf{0}$$

$$\boldsymbol{\pi}_\tau^\times(\mathbf{E}) = \mathbf{0}, \text{ en } \Gamma_\text{D}$$

$$\boldsymbol{\pi}_\tau^\times\left(\mu_r^{-1}(\boldsymbol{\nabla} \times \mathbf{E})\right) = \mathbf{0}, \text{ en } \Gamma_\text{N}$$

$$\boldsymbol{\pi}_\tau^\times\left(\mu_r^{-1}(\boldsymbol{\nabla} \times \mathbf{E})\right) - \gamma_\text{W} \boldsymbol{\pi}_\tau(\mathbf{E}) = \boldsymbol{\Psi}_\text{W}, \text{ en } \Gamma_\text{W}$$

Find $\mathbf{E} \in \boxed{\mathbf{H}_0(\text{curl}, \Omega)}$ such as

$$\left(\boldsymbol{\nabla} \times \mathbf{w}, \mu_r^{-1} \boldsymbol{\nabla} \times \mathbf{E}\right)_\Omega - k_0^2 \left(\mathbf{w}, \varepsilon_r \mathbf{E}\right)_\Omega + \gamma_\text{W} \left\langle \boldsymbol{\pi}_\tau^\times(\mathbf{w}), \boldsymbol{\pi}_\tau^\times(\mathbf{E}) \right\rangle_{\Gamma_\text{W}} = -\left\langle \boldsymbol{\pi}_\tau(\mathbf{w}), \boldsymbol{\Psi}_\text{W} \right\rangle_{\Gamma_\text{W}}$$

$$\forall\, \mathbf{w} \in \boxed{\mathbf{H}_0(\text{curl}, \Omega)}.$$

$$\left(\mathbf{w}, \mathbf{v}\right)_\Omega = \int_\Omega \mathbf{w}^* \cdot \mathbf{v} \, d\Omega,$$

$$\left\langle \mathbf{w}, \mathbf{v} \right\rangle_\Gamma = \int_\Gamma \mathbf{w}^* \cdot \mathbf{v} \, d\Gamma,$$

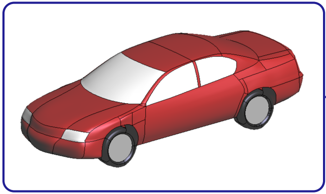
$$\boldsymbol{\pi}_\tau(\mathbf{w}) = \hat{\mathbf{n}} \times (\mathbf{w} \times \hat{\mathbf{n}}) \text{ en } \Gamma,$$

$$\boldsymbol{\pi}_\tau^\times(\mathbf{w}) = \hat{\mathbf{n}} \times \mathbf{w} \text{ en } \Gamma,$$

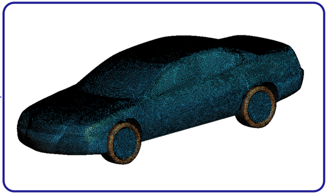
$$\mathbf{H}_0(\text{curl}, \Omega) := \left\{ \mathbf{w} \in [L_2(\Omega)]^3 \,\middle|\, \nabla \times \mathbf{w} \in [L_2(\Omega)]^3 \right\}$$

Basis functions

Introduction



Discretization



$$\nabla \times (\nabla \times \mathbf{E}) - \omega^2 \mu \varepsilon \mathbf{E} = \mathbf{0}$$

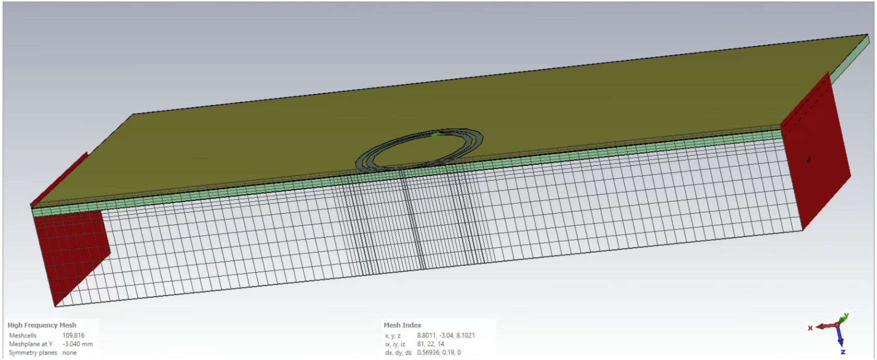
$$\left(\nabla \times \mathbf{w}, \mu_r^{-1} \nabla \times \mathbf{E}\right)_{\Omega} - k_0^2 \left(\mathbf{w}, \varepsilon_r \mathbf{E}\right)_{\Omega} + \gamma_{\mathrm{W}} \left\langle \boldsymbol{\pi}_\tau^\times(\mathbf{w}), \boldsymbol{\pi}_\tau^\times(\mathbf{E}) \right\rangle_{\Gamma_{\mathrm{W}}} = -\left\langle \boldsymbol{\pi}_\tau(\mathbf{w}), \boldsymbol{\Psi}_{\mathrm{W}} \right\rangle_{\Gamma_{\mathrm{W}}}$$

$$\tilde{\mathbf{E}} = \sum_i^n g_i \cdot \boxed{\mathbf{w}_i(x, y, z)} = \mathbf{g}\mathbf{w}$$

$$\mathbf{A}\mathbf{g} = \mathbf{b}$$

$$\boxed{A_{ij}^{(e)} = \int_{\Omega_e} \nabla \times \mathbf{w}_i \cdot \mu_r^{-1} \nabla \times \mathbf{w}_j d\Omega_e + \int_{\Omega_e} \mathbf{w}_i \cdot \varepsilon_r \mathbf{w}_j d\Omega_e + \gamma_{\mathrm{W}} \int_{\Gamma_e} \boldsymbol{\pi}_\tau^\times(\mathbf{w}_i) \cdot \boldsymbol{\pi}_\tau^\times(\mathbf{w}_j) d\Gamma_e}$$

Anisotropic Nédélec Curl-Conforming Prismatic Element



FE Classification

$$\mathbf{H}(\text{curl}, \Omega) := \left\{ \mathbf{w} \in [L_2(\Omega)]^3 \,\middle|\, \boldsymbol{\nabla} \times \mathbf{w} \in [L_2(\Omega)]^3 \right\}$$

Anisotropic Nédélec Curl-Conforming Prismatic Element

- *Curl-conforming* basis functions:
  - Mixed-order
  - Full-order

$$H^1(\Omega) := \left\{ v \in L_2(\Omega) \,\middle|\, \boldsymbol{\nabla} v \in [L_2(\Omega)]^3 \right\}$$

$$\forall\, v_p \in H^1(\Omega), \boldsymbol{\nabla} v_{p+1} \notin \mathbf{w}_p$$

- Jean-Claude Nédélec. “Mixed Finite Elements in R3”. In: *Numerische Mathematik* (1980)
- Jean-Claude Nédélec. “A New Family of Mixed Finite Elements in R3”. In: *Numerische Mathematik* (1986)

$$\mathbf{H}(\text{curl},\Omega) := \left\{\mathbf{w} \in [L_2(\Omega)]^3 \,\middle|\, \boldsymbol{\nabla} \times \mathbf{w} \in [L_2(\Omega)]^3\right\}$$

Anisotropic Nédélec Curl-Conforming Prismatic Element

- *Curl-conforming* basis functions:
  - Interpolatory
  - Hierarchical

$$\mathcal{W}_p = \text{span}(\mathbf{w}_p)$$

$$\mathbf{w}_{p-1} \subset \mathbf{w}_p$$

Why anisotropic?

Anisotropic Nédélec Curl-Conforming Prismatic Element

- Adrian Amor-Martin, Luis E. Garcia-Castillo, and Daniel Garcia-Donoro. "Second-Order Nédélec Curl-Conforming Prismatic Element for Computational Electromagnetics". In: *IEEE Transactions on Antennas and Propagation* (2016)
- Adrian Amor-Martin and Luis E. Garcia-Castillo. "Second-Order Nédélec Curl-Conforming Hexahedral Element for Computational Electromagnetics". In: *IEEE Transactions on Antennas and Propagation* (2023)

Why anisotropic?

Anisotropic Nédélec Curl-Conforming Prismatic Element

- Similar to spectral elements...
- But with Nédélec's mathematical properties.

Basis functions

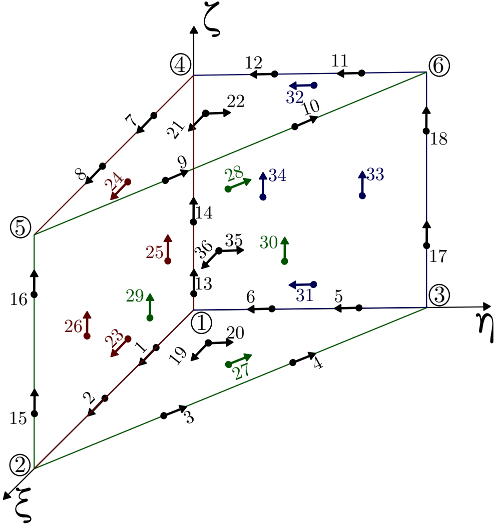
What is a Finite Element?

- Finite Element
  - Domain
  - Space of functions
  - Degrees of freedom

Philippe G Ciarlet. "The Finite Element Method for Elliptic Problems". In: *Classics in applied mathematics* (2002)

Domain

Basis functions



(a) Triangular prism

Construction of the second-order space of functions

- Tensor product between triangle and segment.

$$\mathcal{P}_k = (\mathcal{R}_k(\hat{T}) \otimes \mathcal{P}_k(\hat{I})) \times (\mathcal{P}_k(\hat{T}) \otimes \mathcal{P}_{k-1}(\hat{I}))$$

- Space dimension.

Dimensions	$\mathcal{R}^k(\hat{T})$	$\mathcal{P}_k(\hat{I})$	$\mathcal{P}_k(\hat{T})$	$\mathcal{P}_{k-1}(\hat{I})$	Total
k	$(k+2)k$	$k+1$	$\frac{(k+1)(k+2)}{2}$	k	—
$k=1$	3	2	3	1	9
$k=2$	8	3	6	2	36
$k=3$	15	4	10	3	90
$k=4$	24	5	15	4	180

Construction of the anisotropic space of functions

- Tensor product between k-th order triangle and $k-1$-th order segment.

$$\mathcal{P}_{k,k-1} = (\mathcal{R}_k(\hat{T}) \otimes \mathcal{P}_{k-1}(\hat{I})) \times (\mathcal{P}_k(\hat{T}) \otimes \mathcal{P}_{k-2}(\hat{I}))$$

- Space dimension.

Dimensions	$\mathcal{R}^k(\hat{T})$	$\mathcal{P}_{k-1}(\hat{I})$	$\mathcal{P}_k(\hat{T})$	$\mathcal{P}_{k-2}(\hat{I})$	Total
$k=2$	8	2	6	1	22

Second-order space of functions

Basis functions

$$\boldsymbol{\mathcal{P}}_2 = \left\{ \begin{array}{c} a_1^{(i)} + a_2^{(i)}\xi + a_3^{(i)}\eta + a_4^{(i)}\zeta + a_5^{(i)}\xi\zeta + a_6^{(i)}\eta\zeta + a_7^{(i)}\zeta^2 + a_8^{(i)}\xi\zeta^2 + \\ a_9^{(i)}\eta\zeta^2 + C^{(i)}\eta^2 + D^{(i)}\xi\eta + E^{(i)}\eta^2\zeta + F^{(i)}\xi\eta\zeta + G^{(i)}\eta^2\zeta^2 + H^{(i)}\xi\eta\zeta^2 \\ \\ b_1^{(i)} + b_2^{(i)}\xi + b_3^{(i)}\eta + b_4^{(i)}\zeta + b_5^{(i)}\xi\zeta + b_6^{(i)}\eta\zeta + b_7^{(i)}\zeta^2 + b_8^{(i)}\xi\zeta^2 + \\ b_9^{(i)}\eta\zeta^2 - C^{(i)}\xi\eta - D^{(i)}\xi^2 - E^{(i)}\xi\eta\zeta - F^{(i)}\xi^2\zeta - G^{(i)}\xi\eta\zeta^2 - H^{(i)}\xi^2\zeta^2 \\ \\ c_1^{(i)} + c_2^{(i)}\xi + c_3^{(i)}\eta + c_4^{(i)}\xi^2 + c_5^{(i)}\eta^2 + c_6^{(i)}\xi\eta + c_7^{(i)}\zeta + c_8^{(i)}\xi\zeta + \\ c_9^{(i)}\eta\zeta + c_{10}^{(i)}\xi^2\zeta + c_{11}^{(i)}\eta^2\zeta + c_{12}^{(i)}\xi\eta\zeta \end{array} \right\}$$

Anisotropic space of functions

$$\boldsymbol{\mathcal{P}}_{2,1} = \left\{ \begin{array}{c} a_1^{(i)} + a_2^{(i)}\xi + a_3^{(i)}\eta + a_4^{(i)}\zeta + a_5^{(i)}\xi\zeta + a_6^{(i)}\eta\zeta + \dots \\ \dots + C^{(i)}\eta^2 + D^{(i)}\xi\eta + E^{(i)}\eta^2\zeta + F^{(i)}\xi\eta\zeta \\ \\ b_1^{(i)} + b_2^{(i)}\xi + b_3^{(i)}\eta + b_4^{(i)}\zeta + b_5^{(i)}\xi\zeta + b_6^{(i)}\eta\zeta + \dots \\ \dots - C^{(i)}\xi\eta - D^{(i)}\xi^2 - E^{(i)}\xi\eta\zeta - F^{(i)}\xi^2\zeta \\ \\ c_1^{(i)} + c_2^{(i)}\xi + c_3^{(i)}\eta + c_4^{(i)}\xi^2 + c_5^{(i)}\eta^2 + c_6^{(i)}\xi\eta \end{array} \right\}$$

Second-order DOFs

- Edges,

$$g(\mathbf{u}) = \int_e (\mathbf{u} \cdot \hat{\boldsymbol{\tau}}) q \, dl, \forall q \in P_1(e).$$

- Triangular faces,

$$g(\mathbf{u}) = \int_{f_t} (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{q} \, ds, \forall \mathbf{q} \in \mathcal{P}_0(f_t).$$

- Quadrilateral faces,

$$g(\mathbf{u}) = \int_{f_q} (\mathbf{n} \times \mathbf{u}) \cdot \mathbf{q} \, ds, \forall \mathbf{q} = (q_1, q_2); q_1 \in \mathcal{Q}_{0,1}; q_2 \in \mathcal{Q}_{1,0}.$$

- Volume,

$$g(\mathbf{u}) = \int_v \mathbf{u} \cdot \mathbf{q} \, dV, \forall \mathbf{q} \in \mathcal{P}_0.$$

New DOFs

Basis functions

- Horizontal edges (2 per edge),

$$g(\mathbf{u}) = \int_e (\mathbf{u} \cdot \hat{\boldsymbol{\tau}}) q \, dl, \forall \, q \in P_1(e).$$

- Vertical edges (1 per edge),

$$g(\mathbf{u}) = \int_e (\mathbf{u} \cdot \hat{\boldsymbol{\tau}}) q \, dl, \forall \, q \in P_0(e).$$

- Triangular faces (2 per face),

$$g(\mathbf{u}) = \int_{f_t} (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{q} \, ds, \forall \, \mathbf{q} \in \mathcal{P}_0(f_t).$$

- Quadrilateral faces (1 per face),

$$g(\mathbf{u}) = \int_{f_q} (\mathbf{n} \times \mathbf{u}) \cdot \mathbf{q} \, ds, \forall \, \mathbf{q} = (q_1, 0); q_1 \in \mathcal{P}_0.$$

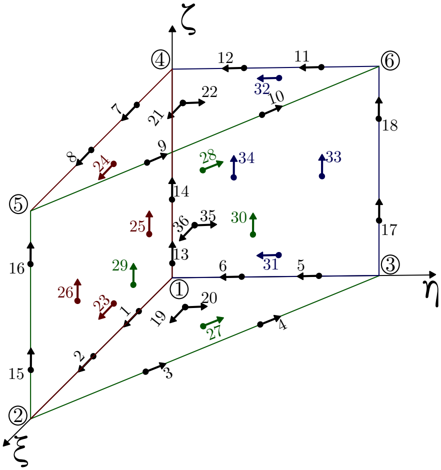
Dual bases

- Dual bases using the polynomials that compose the space,

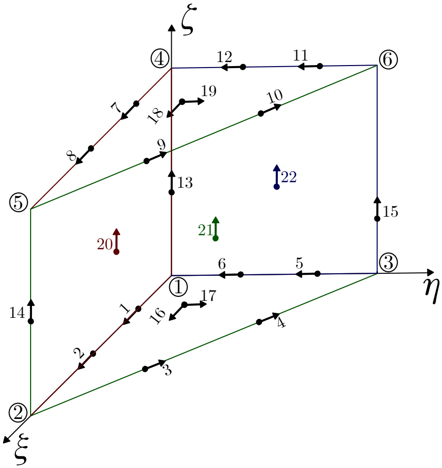
$$g_i(\mathbf{w_j}) = \delta_{ij} \rightarrow \mathbf{gc} = \mathbb{I}.$$

$$\begin{cases} a_1^{(i)} g_i([1,0,0]) + \ldots + D^{(i)} g_i([\xi\eta, \xi^2, 0]) + \ldots + c_{12}^{(i)} g_i([0,0,\xi\eta\zeta]) = 1 \\ a_1^{(j)} g_i([1,0,0]) + a_2^{(j)} g_i([\xi,0,0])0 + \ldots + c_{12}^{(j)} g_i([0,0,\xi\eta\zeta]) = 0 \\ a_1^{(i)} g_j([1,0,0]) + \ldots + b_4^{(i)}([0,\zeta,0]) + \ldots + c_{12}^{(i)} g_j([0,0,\xi\eta\zeta]) = 0 \end{cases}$$

Distribution of DOFs



(a) Second-order prism



(b) New prism

q discretization

Basis functions

DOF	q	DOF	$\mathbf{q}$
$1,7$	$1-\xi$	$16,18$	$\boldsymbol{\xi}$
$2,8$	ξ	$17,19$	$\boldsymbol{\eta}$
$3,9$	ξ	$20,21,22$	$\boldsymbol{\zeta}$
$4,10$	η	$-$	$-$
$5,11$	η	$-$	$-$
$6,12$	$1-\eta$	$-$	$-$
$13,14,15$	1	$-$	$-$

Results

Results

Involved matrices

- Mass matrix M

$$M_{ij} = \int_{\Omega} \mathbf{w}_i \cdot \mathbf{w}_j d\Omega$$

- Stiffness matrix K

$$K_{ij} = \int_{\Omega} (\nabla \times \mathbf{w}_i) \cdot (\nabla \times \mathbf{w}_j) d\Omega$$

Tests on the reference element

- Mass matrix is positive definite.
- Stiffness matrix is positive semi-definite:
  - $\#(\lambda = 0) = \text{size}\left(H^1(\Omega)\right) - 1$
  - Tensor-product: $\text{size}\left(H^1(\Omega)\right) = 12$.
- Extended matrix has 22 null eigenvalues.

$$M_{\text{extended}} = \begin{bmatrix} M_{\text{polynomials}} & M_{\text{polynomials-bases}} \\ M_{\text{bases-polynomials}} & M_{\text{bases}} \end{bmatrix}$$

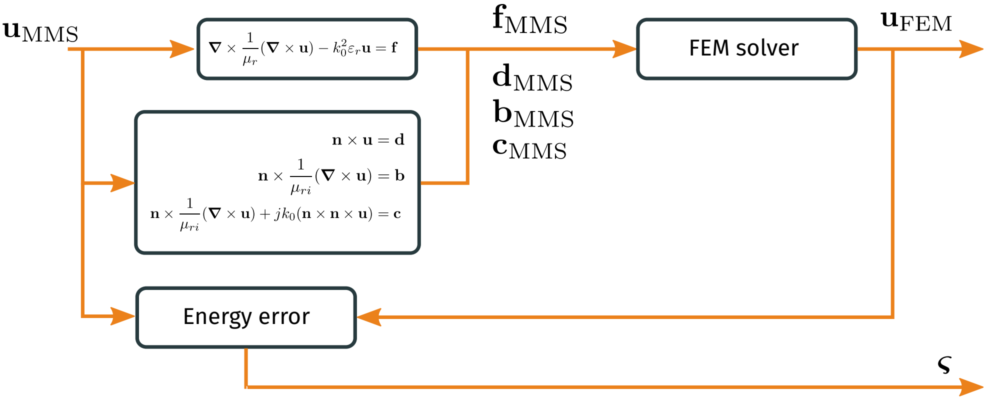
Condition number

$$M_p = D_M^{-1} M D_M^{-1}, \; D_{M,ii} = \sqrt{M_{ii}}$$

$$K_p = D_K^{-1} K D_K^{-1}, \; D_{K,ii} = \sqrt{K_{ii}}$$

	M_p	K_p
New prism	55	25
Prism $p = 1$	9	5
Prism $p = 2$	82	44

Method of Manufactured Solutions



Adrian Amor-Martin, Luis E. Garcia-Castillo, Laszlo L. Toth, Oliver Floch, and Romanus Dyczij-Edlinger. “A Rigorous Code Verification Process of the Domain Decomposition Method in a Finite Element Method For Electromagnetics”. In: *IEEE Transactions on Antennas and Propagation* (2024)

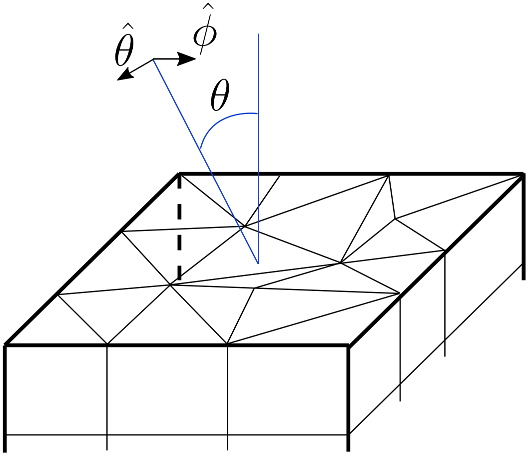
Monomials test

Monomial	ς	ς_∇	Monomial	ς	ς_∇
$[1,0,0]$	$1.1e-15$	$6.3e-15$	$[yz,0,0]$	$1.1e-15$	$1.9e-15$
$[0,1,0]$	$1.0e-15$	$6.2e-15$	$[0,xz,0]$	$1.3e-15$	$2.1e-15$
$[x,0,0]$	$9.7e-16$	$1.5e-15$	$[0,yz,0]$	$1.3e-15$	$2.8e-15$
$[y,0,0]$	$1.3e-15$	$2.9e-15$	$[-xyz,x^2z,0]$	$1.8e-15$	$2.6e-15$
$[0,x,0]$	$1.8e-15$	$3.6e-15$	$[y^2z,-xyz,0]$	$1.7e-15$	$2.5e-15$
$[0,y,0]$	$9.7e-16$	$1.7e-15$	$[0,0,1]$	$4.3e-16$	$1.9e-15$
$[-xy,x^2,0]$	$2.4e-15$	$3.7e-15$	$[0,0,x]$	$7.0e-16$	$1.3e-15$
$[y^2,-xy,0]$	$2.5e-15$	$3.8e-15$	$[0,0,y]$	$7.3e-16$	$1.3e-15$
$[z,0,0]$	$1.2e-15$	$2.3e-15$	$[0,0,xy]$	$1.0e-15$	$1.9e-15$
$[0,z,0]$	$1.1e-15$	$2.2e-15$	$[0,0,x^2]$	$1.4e-15$	$2.4e-15$
$[xz,0,0]$	$1.2e-15$	$2.7e-15$	$[0,0,y^2]$	$1.6e-15$	$2.2e-15$

Convergence: technical details (i)

Case	# Elem.	Unkn. new prism	Unkn. $p = 1$	Unkn. $p = 2$
1	16	98	34	172
2	64	604	192	1120
3	256	4184	1264	8032
4	1024	31024	9120	60736
5	4096	238688	69184	472192

Convergence: technical details (ii)

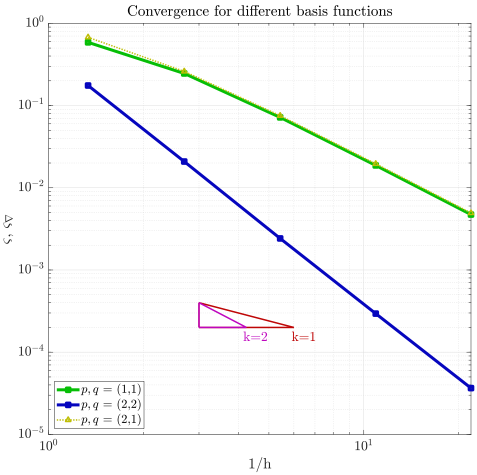


- Smooth solution as manufactured solution.

$$\mathbf{V}_{\mathrm{MMS}} = \mathbf{V}_{\mathrm{pol}} e^{-jk_0(\mathbf{k_p}\cdot\mathbf{r})},$$

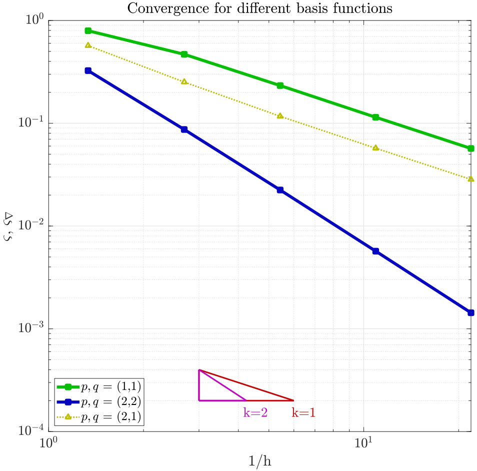
Convergence results (i)

$\theta = 0, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\theta}}$



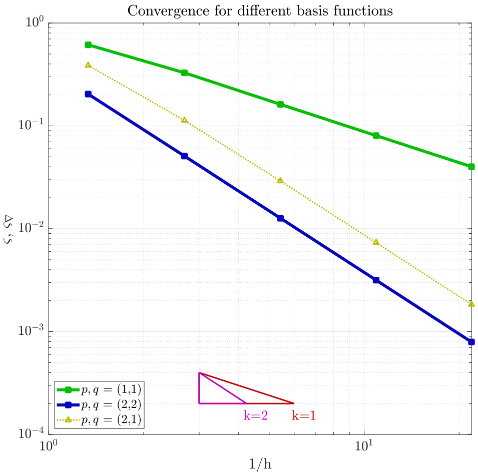
Convergence results (ii)

$$\theta = \frac{\pi}{4}, \mathbf{V}_{\mathrm{pol}} = \widehat{\boldsymbol{\theta}}$$



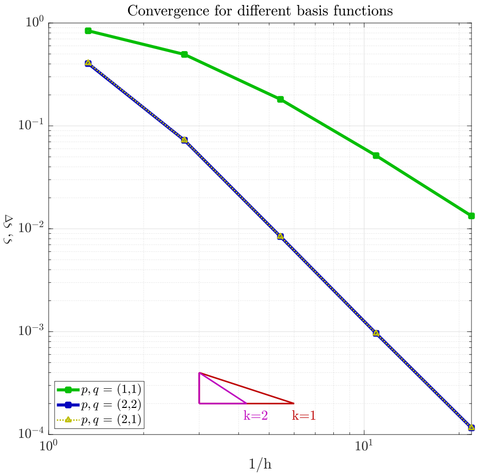
Convergence results (iii)

$\theta = \frac{\pi}{4}, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\phi}}$

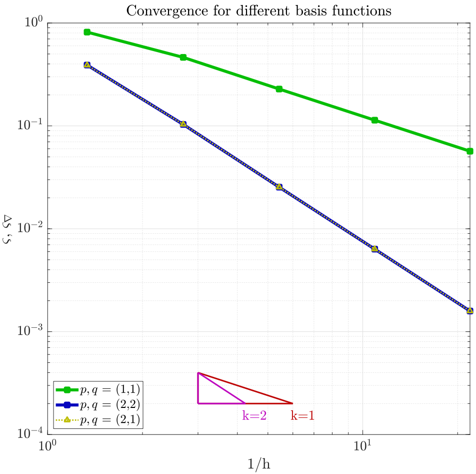


Convergence results (iv)

Superconvergence: $\theta = \frac{\pi}{2}, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\theta}}$

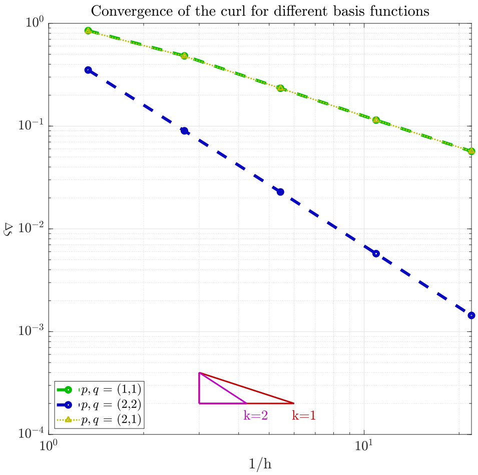


$\theta = \frac{\pi}{2}, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\phi}}$



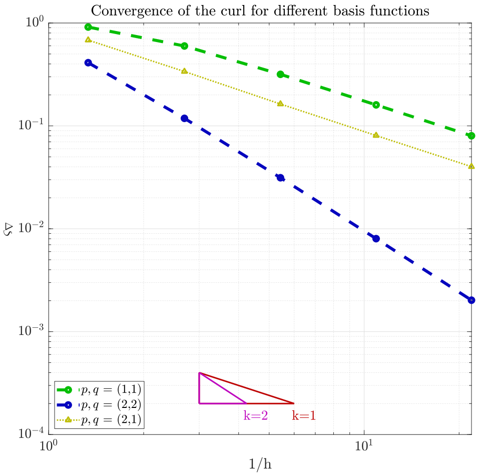
Curl Convergence results (i)

$\theta = 0, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\theta}}$



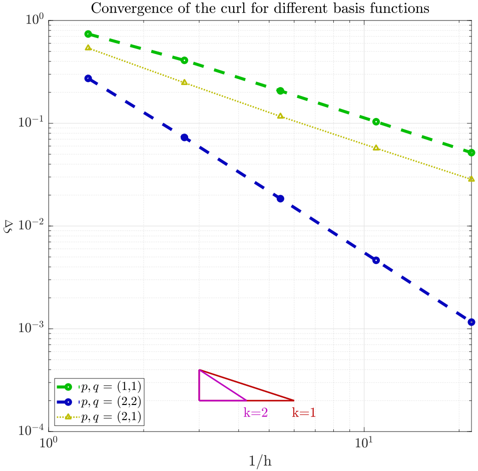
Curl Convergence results (ii)

$\theta = \frac{\pi}{4}, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\theta}}$



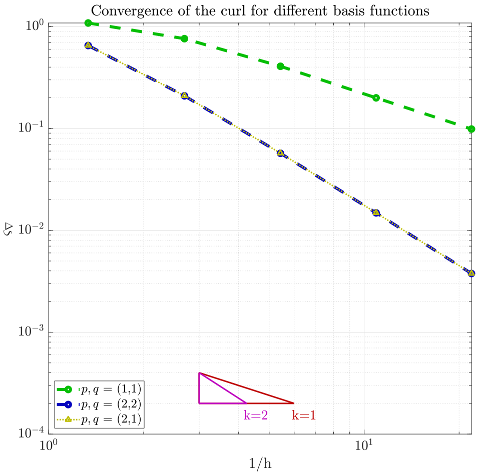
Curl Convergence results (iii)

$\theta = \frac{\pi}{4}, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\phi}}$



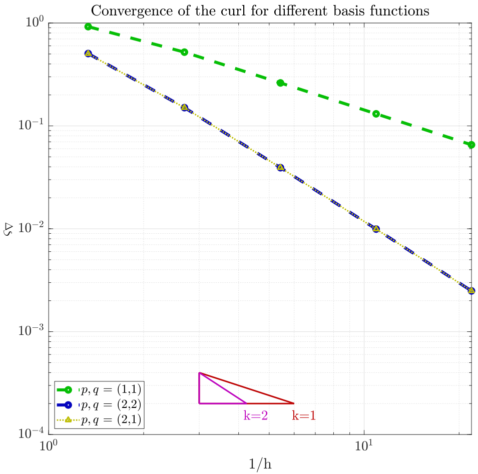
Curl Convergence results (iv)

$$\theta = \frac{\pi}{2}, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\theta}}$$



Curl Convergence results (v)

$$\theta = \frac{\pi}{2}, \mathbf{V}_{\text{pol}} = \widehat{\boldsymbol{\phi}}$$



Conclusions and future lines

Conclusions

- ✓ Systematic approach for anisotropic Nédélec curl-conforming prismatic element.
- ✓ Verification (MMS) and validation of the implementation.
- ✓ Good condition number without orthogonalization strategies.
- ✓ Convergence between uniform first and second-order elements.

Future lines

- ✓ Comparison in terms of accuracy and performance with real problems.
- Extrusion of tetrahedral mesh.

Adrián Amor-Martín
CMMSE 2025